DIAGNOSING EXAMINEES' ATTRIBUTES-MASTERY USING THE BAYESIAN INFERENCE FOR BINOMIAL PROPORTION: A NEW METHOD FOR COGNITIVE DIAGNOSTIC ASSESSMENT

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DEDICATION

I dedicate this paper to the true author, the Lamb of God, who provided a bit of His wisdom from the beginning to the end of this study.

"Truly my soul silently waits for God; From Him comes my salvation. He only is my rock and my salvation; He is my defense; I shall not be greatly moved." (Psalms 62:1-2)

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SUMMARY

Cognitive diagnostic assessment (CDA) is a new theoretical framework for psychological and educational testing that is designed to provide detailed information about examinees' strengths and weaknesses in specific knowledge structures and processing skills. During the last three decades, more than a dozen psychometric models have been developed for CDA, which are also called cognitive diagnosis models (CDM). Although they have successfully provided useful diagnostic information about the examinee, most CDMs are complex due to a large number of parameters in proportion to the number of skills (attributes) to be measured in an item. The large number of parameters causes heavy computational demands for the estimation. Also, a variety of specific software applications is needed depending on the chosen models.

Purpose of this study was to propose a simple and effective method for CDA without heavy computational demand using a user-friendly software application. Bayesian inference for binomial proportion (BIBP) was applied to CDA because of the following fact: When we have binomial observations such as item responses (right/wrong), using a *beta* distribution as a prior of a parameter to estimate (i.e., attribute-mastery probability) makes it very simple to find the *beta* posterior of the parameter without any integration. The application of BIBP to CDA can be flexible depending on the test item-attribute design and examinees' attribute-mastery patterns. In this study, effective ways of applying the BIBP method was explored using real data studies and simulation studies. Also, other preexisting diagnosis models such as DINA and LCDM were compared to the BIBP method in their diagnosis results.

In real data studies, the BIBP method was applied to a test data using two different item designs: four and ten attributes. Also, the BIBP method was compared with DINA and LCDM in their diagnosis result using the same four-attribute data set. There were slight differences in the attribute mastery probability estimate ($\hat{\pi}_k$) among the three model (DINA, LCDM, BIBP), which could result in different diagnosis results for attribute mastery pattern (α_k). Simulation studies were conducted to (1) evaluate general accuracy of the BIBP parameter estimation, (2) examinee the impact of various factors such as attribute correlation (no, low, medium, and high), attribute difficulty (easy, medium, and hard) and sample size (100, 300, and 500) on the consistency of the parameter estimation of BIBP, and (3) compare the BIBP method with the DINA model in the accuracy of recovering true parameters. It was found that the general accuracy of the BIBP method in the true parameter estimation was relatively high. The DINA estimation showed slightly higher overall correct classification rate but the bigger overall biases and estimation errors than the BIBP estimation. The three simulation variables (Attribute Correlation, Attribute Difficulty, and Sample Size) showed significant impacts on the parameter estimations of both models. However, they affected differently the two models: Harder attributes showed the higher accuracy of attribute mastery classification in the BIBP estimation whereas easier attributes were associated with the higher accuracy of the DINA estimation. In conclusion, BIBP appears an effective method for CDA with the advantage of easy and fast computation and a relatively high accuracy of parameter estimation.

CHAPTER 1

INTRODUCTION

1.1 What is Cognitive Diagnostic Assessment (CDA)?

Cognitive diagnostic assessment (CDA) is a new theoretical framework for psychological and educational tests designed to diagnose examinees' skill profiles rather than just rank the examinees based on test scores. Although traditional tests have served to grade and rank examinees' test performance successively, they do not typically provide useful diagnostic information about each examinee (Chipman, Nichols, & Brennan, 1995). CDA does provide detailed information about examinees' strengths and weaknesses in specific knowledge structure and processing skills so that examinees can understand why they pass or fail in a specific item and improve their future performance.

The history of CDA can be traced to the late 1960s and early 1970s when *cognitive psychology* (which examines internal mental processes including how people think, perceive, remember and learn) and *psychometrics* (which is concerned with the theories, models and statistical techniques applied to develop psychological and educational tests) met (Bejar, 2008; Chipman et al., 1995; Leighton & Gierl, 2007). During that time, Item Response Theory (IRT: Lord & Novick, 1968; Rasch, 1960) and its psychometric models (i.e., Rasch, 2PL, 3PL IRT models) emerged and IRT became the mainstream of modern psychometric theory.

Cognitive psychology has provided a much improved understanding of the component processing skills, strategies, and knowledge structures underlying the test performance that is a key part of CDA (Snow & Lohman, 1989). Specifically,

information-processing analysis of problem solving which began in the 1970s (e.g., Carroll, 1976; Hogaboam & Pellegrino, 1978; Hunt, Frost, & Lunneborg, 1973; Sternberg, 1977) has greatly influenced important developments in CDA. According to Mislevy (2006), the focus of the information-processing analysis is on "what's happening within people's heads" (p. 262) while they are responding to items. As Bejar (2008) noted, many studies about cognitive ability such as analogical reasoning (Sternberg, 1977), spatial reasoning (Egan, 1979; Pellegrino & Kail, 1982), inductive reasoning (Pellegrino & Kail, 1982), and verbal ability (Hunt, 1978; Hunt, Lunnenborg, & Lewis, 1975) help psychometricians understand a wide variety of cognitive skills underlying test performance. Thus, the interpretation of test performance reflects a complex combination of component processing skills, strategies, and knowledge structures (Snow& Lohman, 1993).

The synergy between cognitive psychology and psychometrics also led to cognitively based item generation in the 1980's (e.g., Bejar, 1985; Butterfield, Nielsen, Tangen, & Richardson, 1985; Bejar & Yocom, 1986; Hornke, 1986). As noted by Bejar (2008), Embretson's publications (1983, 1985) "created momentum for the idea of cognitively based item generation" (p. 11). Cognitively based item generation implies that a well-developed cognitive theory can provide a framework for guiding item selection and design. Snow and Lohman (1989) and Messick (1989) were inspired by several precedent research studies (e.g., Cronbach, 1970; Embretson, 1983; Pellegrino & Glaser, 1979) and escalated interest in and emphasized the need for cognitive diagnostic assessment in their book chapters in 1989. Since then, the term, CDA, has been widely

used and has grown into one of the major issues in development of ability and achievement tests (Leighton & Gierl, 2007).

For the last two or three decades, researchers have proposed several psychometric models and test design approaches to implement CDA. Of the psychometric models, the linear logistic test model (LLTM; Fischer, 1973) is considered to be the first psychometric model which effectively bridged cognitive psychology and psychometrics. In the 1980s, the multidimensional latent trait model (MLTM; Whitely, 1980; Embretson, 1991) and the general component latent trait model (GLTM; Embretson, 1984) were developed as multidimensional and noncompensatory extensions of the Rasch model and the linear logistic test model, respectively. At that time, the rule space methodology (K. K. Tatsuoka, 1983, 1990; K. K. Tatsuoka & M. M. Tatsuoka, 1987) was also proposed and became a cornerstone of many diagnostic models in educational measurement (e.g., unified model, fusion model, DINA, AHM). In the 1990's, several influential studies about test design approach and psychometric models for CDA were introduced such as the cognitive design system (Embretson, 1992, 1994), Evidence-Centered Design (Mislevy, 1994), unified model (DiBello, Stout, & Roussos, 1993), fusion model (Hartz, 2002; Roussos et al., 2007), Deterministic Inputs, Noisy "And" gate (DINA) model (Haertel, 1989), Noisy Inputs, Deterministic "And" gate (NIDA) model (Junker & Sijtsma, 2001), Deterministic Inputs, Noisy "Or" gate (DINO) model (Templin & Henson, 2006), HYBRID model (Yamamoto, 1989), 2PL-Constrained model (Embretson, 1999), and log-linear cognitive diagnostic model (LCDM; Henson, Templin, & Willse, 2009).

1.2 Current Issues in CDA

Although the psychometric models for CDA, which are also called cognitive diagnosis models (CDM) or diagnostic classification models (DCM), have own strengths and weaknesses, they successfully provided useful diagnostic information about the examinee, as well as about each test item. However, most CDMs are complex due to a large number of parameters in proportion to the number of skills to be measured in an item. For example, fusion model has k+2 parameters (k = number of skills to be measured in an item) and the DINO model includes 2k parameters to be estimated. The large number of parameters in the models cause heavy computational demands for the parameter estimation. In some models such as fusion model, Markov chain Monte Carlo (MCMC) method is used for the parameter estimation since it is easier to extend to parametrically complex models than Expectation Maximization (EM) algorithms. However, the MCMC causes even heavier computational demand than the marginal maximum likelihood estimation (MMLE) with the EM algorithm. It takes several hours even for a single estimation and a day or more for more complex models or large amounts of data. Also, the MCMC can be misused easily because of the complexity of its algorithms, thus, it is uncommon for users to take the result of the MCMC analysis with confidence (Kim & Bolt, 2007).

A variety of software applications is needed depending on the chosen models for the parameter estimation. Some examples of typical software applications for the CDMs include (Rupp & Templin, 2008):

- Rule space method: BUGLIB (Research License, tatsuoka@prodigy.net)
- Fusion model: Arppegio (Commercial, www.assess.com)

- DINA, NIDA, and DINO: DCM (Free ware but requires the commercial version of Mplus, jtemplin@uga.edu)
- DINA and DINO: DCM in R (Free ware requiring the freeware R, alexander.robitzsch@iqb.hu-berlin.de)
- LLTM: ConQuest (Commercial, www.assess.com), LPCM-WIN (Commercial, www.assess.com), SAS (Commercial, www.sas.com)
- LPCM: LPCM-WIN (Commercial, www.assess.com)
- 2PL-Constrained model, MLTM, & GLTM: SAS (Commercial, www.sas.com)

It is still challenging for general users to run most of these software packages with confidence because they are complex to operate and uncertainty exists about the analysis results, especially for a complex model with heavy computational demands.

1.3 Purpose of the Study

Purpose of this study was to propose a simple and effective method for CDA without heavy computational demand and using an user-friendly software application. Bayesian inference for binomial proportion (BIBP) was applied to CDA because of the following fact: When we have binomial observations such as item responses (right/wrong), using a *beta* distribution as a prior of a parameter to estimate (i.e., attribute-mastery probability) makes it very simple to find the *beta* posterior of the parameter without any integration in the Bayesian framework (Bolstad, 2007). Therefore, first, how to effectively apply BIBP to CDA was explored using a real test data with different item designs. The diagnosis results (e.g., examinees' attribute mastery probability and the patterns) of the BIBP method were also compared to the results of

other diagnosis models. Second, using simulated data, the accuracy of the parameter estimation of the BIBP method was evaluated and how various conditions on examinee and item design can affect the estimation accuracy was also explored.

CHAPTER 2

Cognitive Diagnosis Model (CDM)

2.1 Overview

Although several different ways of classifying cognitive diagnosis models may exist, they were divided into two categories in this study: (1) *latent class model* and (2) *latent trait model*. A latent class model denotes the model which classifies examinees into categories on a set of skills (e.g., mastery or nonmastery of the skill), thus providing a mastery pattern (or mastery probabilities) as the examinee's skill profile. Fusion model, DINA, NIDA, and DINO models are in this category (Roussos et al., 2008). Whereas, a latent trait model is an extension or a generalization of unidimensional IRT models (e.g., Rasch, 2PLM) that place estimate examinee's ability on a continuous scale for each skill; LLTM, MLTM, GLTM, and 2PL-constrained model belong to this category. The list of the CDMs for the two categories is presented in Table 2.1.

In the latent class model-category, the rule space method was a pioneering and successful method of diagnosing examinees' knowledge levels as an attempt to overcome the limitation of the traditional test scoring system where valuable information from the examinee's item response pattern is thrown away for the sake of simplicity. It has greatly influenced the development of all the subsequent models in this category (e.g., DINA, Unified, Fusion). The rule space method is not a single psychometric model, but rather a system which guides both how to diagnose an examinee's skill profile and how to improve the examinee's knowledge level. However, the rule space method has a limitation in treating a variety of sources of response variation which caused mismatches

between observed and Q-predicted response patterns. The contribution of the unified model was to find a way of identifying and treating different sources of response variation such as Strategy Selection, Completeness, Positivity, and Slips. Then, the Fusion model was developed to overcome the limited identifiability of the unified model while maintaining its flexibility and interpretability.

Table 2.1 *List of the CDMs in Two Categories*

Years	*Latent Class Models*	*Latent Trait Models*
1970		LLTM (Fischer, 1973)
1980	Rule space (K. K. Tatsuoka, 1983) DINA (Haertel, 1989) HYBRID model (Yamamoto, 1989)	MLTM (Embretson, 1980) MIRT-C (McKinley & Reckase, 1982) GLTM (Embretson, 1984)
1990	Unified model (DiBello, Stout, & Roussos, 1993)	LPCM (Fischer & Ponocny, 1994) 2PL-Constrained model (Embretson, 1999)
2000	NIDA (Junker & Sijtsma, 2001) Fusion (Hartz, 2002) AHM (Leighton, Gierl, & Hunka, 2004) DINO (Templin & Henson, 2006) LCDM (Henson, Templin, & Willse, 2009)	MLTM-D (Embretson & Yang, 2008)

The Fusion model is mathematically equivalent to the original unified model but has reduced the number of item parameters from $2k + 3$ to $k+2$ (k = number of skills) by setting the strategy selection parameter to 1. Therefore, the Fusion model traded the flexibility of handling multiple strategies for the identifiability of all item parameters. Yet, Fusion keeps good flexibility to deal with incomplete Q-matrix and positivity of attributes, which provides information about how effective an item is in measuring the

attributes. However, there exist some drawbacks due to the parameter estimation method (MCMC) used in Fusion, which causes heavy computational demand and uncertainty about the analysis result, especially when the number of skills or attributes (*k*) to be measured in an item increases.

An important feature of DINA model and its followers (NIDA, DINO) is the dealing with false positive and false negative errors of examinees, which correspond to *positivity* in the unified model and fusion model. Although they share the same mathematical concept, there are some differences with each other. On one hand, DINA and NIDA are non-compensatory models, thus appropriate for the skill diagnosis where consecutive skills should be successfully performed in order to arrive at a correct answer (e.g., a mathematical test). On the other hand, DINO is a compensatory model which is increasingly applied to the settings like medical and psychological disorder diagnoses. Furthermore, DINA is a more complex model than NIDA because DINA was designed in the item-level perspective while NIDA was constructed in the skill-level perspective and the number of items is always bigger than the number of skills to be measured in a test. The HYBRID model has an interesting characteristic. It was developed to handle response variation caused by the two strategies used by examinees (Q-based strategy and non-Q-based strategy), which correspond to *strategy selection* in the unified model. Based on examinee response patterns, HYBRID model adopts a non-compensatory latent class model (e.g., DINA) for the Q-based strategy and a unidimensional IRT model for the non-Q-based strategy. The log-linear cognitive diagnostic model (LCDM; Henson et al., 2009) is a generalized model to express both compensatory and non-compensatory models.

Although most of those latent class models are very practical and useful to diagnose examinee skill profiles, they cannot estimate direct effects of the skills or item stimulus features on cognitive requirements to solve an item. Thus, they are not useful for a cognitive model-based test design in which cognitive theories are incorporated into item generation. AHM is an exception in this category. It incorporates a cognitive model of structured attributes into the test design, following the approach of information-processing analysis of problem solving. AHM classifies examinee's test performance into a set of structured attribute patterns by the product of probabilities of positive and negative slips (corresponding to false positive and false negative probabilities in DINA) using an unidimensional IRT model. However, if examinees make several slips, then, very low likelihood estimates usually happen, which makes it very hard to classify the examinees into the structured attribute patterns.

In the latent trait model-category, LLTM, 2PL-Constrained model, LPCM, and GLTM allow test developers to incorporate item stimulus features based on cognitive theories and to estimate direct effects of the stimulus features on cognitive requirements for success in the item. Therefore, they met the need of psychometric models for cognitive model-based test design. Latent trait models can be divided into two subcategories which are unidimensional and multidimensional models. Although multidimensional models, such as compensatory multidimensional IRT model (MIRT-C; McKinley & Reckase, 1982), MLTM, GLTM, are appropriate for diagnosing multiple-skill profiles, unidimensional models, such as LLTM, 2PL-Constrained model, linear partial credit model (LPCM; Fischer & Ponocny, 1994), are also very useful for CDA because most achievement tests typically fit unidimensional IRT models fairly well and

these models are simple and easy to apply for the test design. In the multidimensional model-subcategory, MLTM and GLTM are appropriate for most ability and achievement tests because typical skills (attributes) and processing stages in the test items are sequentially dependent, whereas, MIRT-C seems more appropriate for medical and psychological disorder diagnoses like DINO. One drawback of MLTM and GLTM is that these models typically need both subtask and full task data for the same item, which makes the data collecting process complex. Recently, Embretson and Yang (2008) proposed multicomponent latent trait model for cognitive diagnosis (MLTM-D) which was specifically designed for CDA.

In this chapter, some of the popular CDMs were reviewed in more detail. They are Rule space method, Fusion model, DINA, NIDA, DINO, LLTM, MLTM, GLTM, and LCDM.

2.2 Rule Space Method

Tatsuoka and her associates (K. K. Tatsuoka, 1983, 1990; K. K. Tatsuoka & M. M. Tatsuoka, 1987) developed a pioneering method to diagnose examinees' knowledge levels. This method was named *Rule Space Methodology*. Development of the *Rule Space Methodology* was motivated by the issue that traditional test scores have the limitation of providing detailed information about examinees' knowledge structure that underlies test performance. In other words, the same total scores do not necessarily reflect the same level of knowledge or understanding of the examinees. Valuable information from the examinees' item response patterns is thrown away for the sake of simplicity in traditional test scoring. Tatsuoka (1983) believed that analysis of student's misconceptions

throughout a test could provide "specific prescriptions for planning remediation for a student as well as useful information in evaluating instruction or instructional material" (p. 345). The *rule space methodology* includes two parts: (1) *Q-matrix theory* and (2) *rule space*.

First, the Q-matrix is a [$k \times n$] matrix of ones and zeros, in which k represents the number of attributes to be measured and n represents the number of items on the test. For example, there are 12 fraction addition problems in Table 2.2, and nine attributes or cognitive tasks (A_1 through A_9) required to answer the problems (Tatsuoka, 1997). In the bottom of the table, Q-matrix for this test are provided. Because there are twelve items, the dimension of the Q-matrix became [9×12]; nine attributes and 12 items. In the Q-matrix, a one indicates that the item measures the attribute, and a zero indicates that it does not. Therefore, item 1 measures all attributes but 3 and 9. Item 3 measures attributes 4, 6, and 7. All attributes that an item measures should be mastered by an examinee in order to answer the item correctly. If an examinee answered the item 1 correctly, then, theoretically, it means that he/she mastered all attributes but 3 and 9. Furthermore, if an examinee's response pattern for the 12 items was like [0 0 1 0 1 0 1 1 0 1 1 1] (i.e., responding correctly to items 3, 5, 7, 8, 10, 11, and 12 and incorrectly to the rest of items), then, we can infer that the examinee mastered attributes 3, 4, 6 and 7 based on the Q-matrix in the table.

The attribute mastery patterns, which consist of various combinations of the nine attributes (see Table 2.2), are referred to as *knowledge states*. K.K. Tatsuoka and M. M. Tatsuoka (1992) identified 33 knowledge states based on the frequency of the rules that examinees used in solving the fraction addition problems in Table 2.2. Of the 33

knowledge states, 11 of the most common knowledge states are described in Table 2.3. Knowledge state #10 is the mastery state that indicates all answers are correct. On the other hand, knowledge state #21 designates the non-mastery state with all the answers wrong.

Table 2.2 *Attributes and Q-matrix for Fraction Addition Problems*

Fraction addition problems, $a(\frac{b}{c})+d(\frac{e}{f})$			
$I_1.\ 2(\frac{8}{6})+3(\frac{10}{6})$	$I_2.\ \frac{2}{5}+\frac{12}{8}$	$I_3.\ \frac{8}{5}+\frac{6}{5}$	$I_4.\ 2(\frac{1}{2})+4(\frac{2}{4})$
$I_5.\ \frac{1}{2}+1(\frac{10}{7})$	$I_6.\ 3(\frac{5}{7})+4(\frac{6}{7})$	$I_7.\ \frac{3}{5}+\frac{7}{5}$	$I_8.\ \frac{1}{3}+\frac{1}{2}$
$I_9.\ 1(\frac{4}{7})+1(\frac{12}{7})$	$I_{10}.\ \frac{3}{5}+\frac{1}{5}$	$I_{11}.\ \frac{3}{4}+\frac{1}{2}$	$I_{12}.\ 2(\frac{5}{9})+1(\frac{1}{9})$
Attributes			
A_1. Separate the whole number part from the fraction part when $a \neq 0$ or $d \neq 0$. A_2. Separate the whole number part from the fraction part when $a \neq 0$ and $d \neq 0$. A_3. Get the common denominator (CD) when $c \neq f$. A_4. Convert the fraction part before getting the CD. A_5. Reduce the fraction part before getting the CD. A_6. Answer to be simplified. A_7. Add two numerators. A_8. Adjust a whole number part. A_9. Combination of A_2 and A_3 Mixed numbers with $c \neq f$.			
Q-matrix $[9 \times 12]$			
Items (12) Attributes (9) $\begin{bmatrix} 1001010010 \\ 1001110010 \\ 0101100100 \\ 1110101010 \\ 1101000000 \\ 1111111010 \\ 1111111111 \\ 1001110010 \\ 0001100000 \end{bmatrix}$			

Next, an ideal response pattern for each knowledge state needs to be established in order to compare it with observed item response patterns so that the observed response pattern can be classified into one of the 33 knowledge states. Notice that the word "ideal" does not mean a perfect response pattern, but rather perfect fit with a knowledge state. For example, the ideal response pattern of the knowledge state #4 is [1 0 1 0 0 1 1 0 1 1 0 1] because all but attribute 3 are mastered in knowledge state #4 and items #2, 4, 5, 8, and 11 measure attribute 3.

Table 2.3 *The 11 Most Common Knowledge States for Fraction Addition Problems*

	Knowledge states
#4	Cannot get the common denominator (CD) but can do simple fraction addition problems (A_1, A_2, A_4, A_5, A_6, A_7, A_8, and A_9 mastered).
#6	Cannot get CDs for the problems involving mixed number(s) (A_1, A_2, A_3, A_4, A_5, A_6, A_7, and A_8 mastered).
#9	Has problems in simplifying answers to the simplest form (A_1, A_2, A_3, A_5, A_7, A_8, and A_9 mastered).
#10	Mastery state: All attributes are answered correctly (A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, and A_9 mastered).
#11	Can do addition only of two simple fractions (F) when they have the same denominators (A_4, A_6, A_7, A_8, and A_9 mastered).
#16	Cannot get CDs and cannot add two reducible mixed numbers (M). Also has problems with simplification of answers (A_1, A_2, A_7, and A_9 mastered).
#21	Non-mastery state: All attributes are answered incorrectly (no attribute mastered).
#24	Cannot add a mixed number and a fraction. Cannot get CDs. Cannot reduce fraction parts correctly before getting the CDs (A_1, A_2, A_4, A_6, A_7, A_8, and A_9 mastered).
#25	Cannot add the combinations of M and F. Also, cannot get CDs (A_2, A_4, A_5, A_6, A_7, and A_9 mastered).
#26	Does not realize that adding zero to a nonzero number *a* yields *a* itself. That is, does not grasp the Identity Principle, $a + 0 = a$ (A_2, A_3, A_4, A_5, A_6, A_7, A_8, and A_9 mastered).
#33	When adding mixed numbers, adds the fractions correctly but omits the whole number part or gets it wrong due to incorrect simplification of the fraction part. (A_1, A_6, A_7, A_8, and A_9 mastered).

An examinee in state #4 cannot answer correctly those five items but should answer correctly the rest of the 12 items. However, it is very rare to have the observed response patterns perfectly match with the theoretically expected response patterns (ideal response patterns) because examinees most likely do not apply the same erroneous rules consistently over the entire test (Birenbaum & Tatsuoka, 1993; Tatsuoka, 1990). Moreover, various random errors such as careless errors, uncertainty, or distraction (referred to as *slips*, hereafter) cause even more deviations of observed response patterns. Therefore, the *rule space* concept was introduced to handle the problem caused by slips.

Second, as mentioned earlier, after establishing ideal response patterns for the knowledge states, the next step is to compare them with observed examinee's item response pattern, thereby classifying the examinee into one of the knowledge state categories. However, because the observed response patterns are not identical to the ideal response patterns, in general, a method called "rule space" is used to deal with this problem.

Rule space is a graphical representation of the knowledge states for each of the ideal response patterns and observed response patterns (Tatsuoka, 1990). In rule space, the distances between ideal and observed response patterns are measured in order to determine which knowledge state is closest to an observed response pattern as shown in Figure 2.1. The figure illustrates a sample rule space in which four knowledge states (#6, 9, 10, and 16 in Table 2.3) and some hypothetical observed response patterns (marked by "x") are represented. Although knowledge states are unobservable traits and cannot be represented in such a space directly, Tatsuoka (1984) utilized item response theory ability

parameter (θ) and the "atypicality parameter (ζ)" to present them in two-dimensional space.

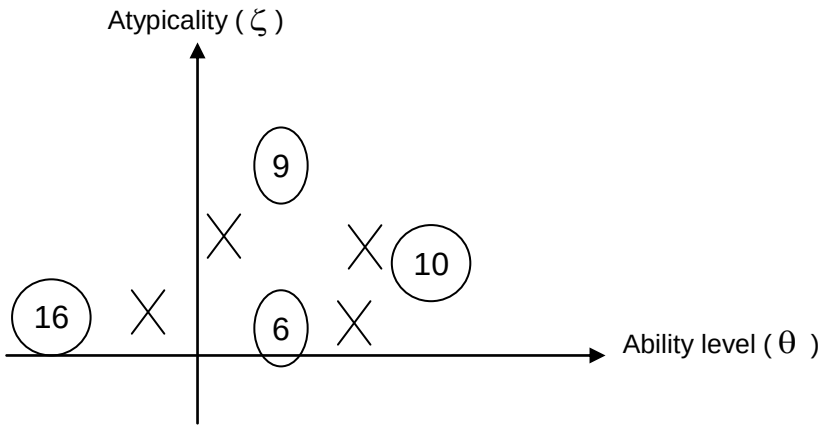


Figure 2.1 Sample rule space of four knowledge states

The value of the atypicality parameter, on the y-axis, indicates how unusual a response pattern is (similar to a person-fit index) and ζ is calculated as the standardized product of two residual matrices between observed and expected values (see Tatsuoka, 1996 for more detail). θ , on the x-axis, represents an examinee's trait or ability estimated by an unidimensional IRT model (e.g., 1PL, 2PL, or 3PL model).

For example, in Figure 2.1, knowledge state #10 (mastery state) requires a higher level of ability than any other states, thereby being the farthest to the right on the θ -axis. State #9 and #6 seem to have same θ value, but different ζ values; state #6 is closer to θ -axis than state #9. This means that state #6 occurs more frequently in the population. Finally, state #16 has a mastery pattern of only four attributes (A1, A2, A7, and A9) and requires a lower level of ability than other knowledge states in Figure 2.1, thus being located at the farthest left on the θ -axis. The distances between the knowledge states

(ideal response patterns) and the observed response patterns can be approximated by the Mahalanobis distance between the centroids of the two points (Tatsuok, 1995). The closest knowledge state to an observed response pattern will be considered the individual's attribute mastery pattern. Bayes' decision rules can also be used to minimize misclassifications because it provides the probability level which attributes a given examinee is likely to have mastered (Birenbaum, Kelly, & Tatsuok, 1993).

The rule space methodology has many advantages over traditional way of assessment. In a traditional test scoring system, individuals within both knowledge state # 6 and #9 could receive the same score. However, the rule space methodology provides more specific information about how their abilities are actually different. Furthermore, rule space illustrates the relationships between the knowledge states by showing how far apart they are using the Mahalanobis distance. Once an examinee is classified into one of the knowledge states, the next knowledge state that the examinee needs to go to will be the closest one to his/her current state. The rule space methodology also provides the way of achieving the next knowledge states. In an adaptive test setting, K. K. Tatsuoka and M. M. Tatsuoka (1997) showed how to provide immediate feedback and remedial instruction to each examinee after diagnosing his/her knowledge state.

In general, the rule space methodology has been considered a practical method of classifying response patterns into knowledge states by simple statistics. Moreover, the Q-matrix theory in this methodology became a foundation for the development of many following diagnostic models (e.g., RUM, Fusion, DINO), especially in educational assessment. However, the rule space methodology has some limitations. There is still uncertainty about how accurately an examinee's knowledge state can be identified given

the variability of possible item response patterns. Also, the rule space methodology is not an approach of a cognitive model-based test design to incorporate cognitive theories into item generation but just a methodology to diagnose students' misconceptions and knowledge states especially in achievement testing.

2.3 Fusion Model (Reparameterized Unified Model)

The unified model (DiBello, Stout, & Roussos, 1993, 1995) has a critical limitation, that is, not all of the parameters in the model can be statistically identifiable. To overcome the limited identifiability while maintaining the advantages of flexibility and interpretability of the unified model, Hartz (2002) reduced the number of parameters from $2k + 3$ to $k+2$ (k = number of skills to be measured in an item). The reduced model is referred to as the reparameterized unified model (RUM), of which all $k+2$ parameters are identifiable. RUM is also called the fusion model. Roussos et al. (2007) defined the *fusion model system* as a CDA system which includes skills diagnosis, the parameter estimation method (i.e., MCMC), model checking procedures and skills-level score statistics. RUM is the item response function model within the fusion model system.

RUM is mathematically equivalent to the original unified model (Roussos et al., 2007). However, there was a trade-off between reducing the number of parameters and a source of flexibility in the original unified model. That is, Strategy Selection (d_i) parameter was omitted in the RUM. Therefore, the probability that an examinee selects a Q-strategy is set to 1 ($d_i = 1$) in the RUM. In other words, there is no possibility that examinees may use other strategies than the Q-strategy to solve the item. If d_i parameter

in the unified model is set to 1, the unified model can be converted into RUM as follows (Roussos et al., 2007):

$$P(X_{ij}=1|\underline{\alpha}_j,\eta_j)=\prod_{k=1}^{K}\pi_{ik}^{\alpha_{jk}\cdot q_{ik}}r_{ik}^{(1-\alpha_{jk})\cdot q_{ik}}p_{c_i}(\eta_j)=\pi_i^*\prod_{k=1}^{K}r_{ik}^{*(1-\alpha_{jk})\cdot q_{ik}}P_{c_i}(\eta_j). \quad (2.1)$$

$P(X_{ij}=1|\underline{\alpha}_j,\eta_j)$ is the probability of answering item *i* correctly given that examinee *j* has a skill mastery vector of $\underline{\alpha}_j$ and a residual (supplemental) ability parameter of η_j, just as in the unified model. π_i^* (= $\prod_{k=1}^{K}\pi_{ik}^{q_{ik}}$) is the probability that an examinee having mastered ALL the skills required for solving item *i* will correctly apply ALL those skills to answer the item. r_{ik}^* is expressed as $\frac{P(Y_{ijk}=1|\alpha_{jk}=0)}{P(Y_{ijk}=1|\alpha_{jk}=1)}=\frac{r_{ik}}{\pi_{ik}}$, where π_{ik} is the probability of applying successfully skill *k* to item *i* given that the examinee has mastered the skill, and r_{ik} is the probability of applying successfully skill *k* to item *i* given that the examinee has NOT mastered the skill.

The r_{ik}^* parameter plays an important role in evaluating the diagnostic ability of an assessment. It distinguishes which item is more effectively discriminating between examinees who have mastered or not mastered skill *k*. For example, if an item more strongly depends on mastery of skill *k*, then the probability of passing the item (r_{ik}) is getting lower for a nonmaster of the skill *k*, thus r_{ik}^* will be close to zero. When the r_{ik}^* parameters are closer to zero for most items of a test, the test will be considered to be well designed for diagnosing mastery on skill *k* (Roussos et al., 2007). This is very similar to the positivity index in the unified model.

The residual ability parameter (η_j) in the $P_{c_i}(\eta_j)$ component was retained from the unified model to deal with the issue that the Q-matrix may not include all necessary skills or attributes for solving an item (incomplete Q-matrix). As in the unified model, $P_{c_i}(\eta_j)$ is the Rasch model with a difficulty parameter of negative c_i (-c_i), which can be expressed as

$$P_{c_i}(\eta_j) = \frac{\exp[\eta_s - (-c_i)]}{1+\exp[\eta_s - (-c_i)]}. \qquad (2.2)$$

It should be noted that if the value of c_i is bigger (meaning an easier item in the Rasch model), then $P_{c_i}(\eta_j)$ will also be higher. In RUM, c_i plays an important role for diagnosing the influence of the missing multiple skills (residual ability) on the whole item response function, $P(X_{ij}=1|\underline{\alpha}_j,\eta_j)$. For example, if c_i is 3 (meaning very easy in the Rasch model), then $P_{c_i}(\eta_j)$ will be almost 1 for most examinees. In such a case, the residual ability (η_j) has almost no impact on $P(X_{ij}=1|\underline{\alpha}_j,\eta_j)$. On the other hand, if c_i is -3, then $P_{c_i}(\eta_j)$ will be close to 0 for most examinees. In such a case, η_j will make $P(X_{ij}=1|\underline{\alpha}_j,\eta_j)$ almost zero regardless of the rest of the parts of $P(X_{ij}=1|\underline{\alpha}_j,\eta_j)$, which indicates that $P_{c_i}(\eta_j)$ has a great impact on $P(X_{ij}=1|\underline{\alpha}_j,\eta_j)$ and that the Q-matrix is incomplete, thus needing to include more skills to be measured.

To estimate item parameters (π_i^*, r_{ik}^*, c_i) and examinee skills parameters ($\underline{\alpha}_j$) in Equation 2.2, the Markov chain Monte Carlo (MCMC) method, which is firmly rooted in the Bayesian inference, has been employed. The MCMC method has several advantages over other parameter estimation methods. First, MCMC algorithms are easier to extend to parametrically complex models such as RUM than Expectation Maximization (EM)

algorithms. Secondly, MCMC provides a joint estimated posterior distribution of both the item parameters and the examinee skills parameters, which provides better understanding of the true standard errors involved (Patz & Junker, 1999). Also, MCMC provides a potentially richer description of the parameters (i.e., a full posterior distribution) than Maximum Likelihood (ML) method which provides an estimate and its standard error because MCMC is based on Bayesian inference on estimating model parameters, (Kim & Bolt, 2007; Roussos, DiBello, Henson, Jang, & Templin, 2008). Finally, a free software is available for MCMC, such as the WINBURGS program (Spiegelhalter, Thomas, Best, & Lunn, 2003), although the Arpeggio program (Hartz, Roussos, & Stout, 2002) is mainly used for parameter estimation of RUM.

MCMC has some disadvantages. The primary drawback of MCMC is the complexity of its algorithms, which causes heavy computational demand and uncertainty about the analysis result in some cases. MCMC algorithms require a large number of iterations until a reliable parameter estimation, thus taking several hours even for a single estimation and a day or more for more complex models or large amounts of data (Kim & Bolt, 2007). Also, MCMC can be misused easily because of the complexity of its algorithms. It is uncommon for users to take the result of the MCMC analysis with confidence, especially with complex models requiring more parameters to estimate (Kim & Bolt, 2007).

The RUM provides useful pieces of information about item properties and examinees' skill profiles while its parameters are identifiable. It estimates each examinee's skill mastery vector $\underline{\alpha}_j$ and a residual (supplemental) ability (η_j). It estimates the item parameter (r_{ik}^*) which evaluates how effectively the item discriminates

between masters and nonmasters of skill *k*. Also, c_i parameter indicates if Q-matrix is incomplete, thus if more skills need to be added in the Q-matrix. These item parameters can be also used to evaluate and support the construct validity of the test. However, the evaluation of convergence for each parameter is difficult because there has yet been no reliable statistic for MCMC convergence check (Roussos et al., 2007) and the RUM is a still complex model having many parameters to estimate.

2.4 DINA, NIDA, and DINO Model

The *Deterministic Inputs, Noisy "And" gate* (DINA; Haertel, 1989) model and the *Noisy Input, Deterministic "And" gate* (NIDA; Junker & Sijtsma, 2001) models are conjunctive (non-compensatory) models for skills diagnosis. The HYBRID model (Yamamoto, 1989) is also a conjunctive model, but, interestingly, is flexible to choose a latent class model such as DINA or an IRT model (1PL, 2PL, or 3PL model) based on examinees' observations. These conjunctive models are appropriate for skill diagnosis where the solution of a task is broken down into a series of steps with conjunctive interaction rather than with compensatory interaction (Roussos et al., 2008). Typical examples of the conjunctive interaction can be found in the skills required to solve mathematical items where the consecutive skills should be successfully performed in order to arrive at a correct answer.

However, the *Deterministic Input; Noisy "Or" gate* (DINO; Templin & Henson, 2006) model is a disjunctive (compensatory) model. The compensatory models have been increasingly applied to a variety of settings, such as medical and psychological disorder diagnosis, where the presence of other symptoms can compensate the absence of certain

symptoms (Rousoss et al., 2008). Those four models (DINA, NIDA, HYBRID, & DINO) are closely related to each other in their item response functions.

2.4.1 DINA Model

The item response function for a single task of the DINA model is

$$P(X_{ij}=1 \mid \underline{\alpha}, s, g) = (1-s_j)^{\xi_{ij}} g_j^{1-\xi_{ij}}, \qquad (2.3)$$

where $\xi_{ij} = \prod_{k=1}^{K} \alpha_{ik}^{Q_{jk}}$ indicates if examinee *i* has all the skills to solve item *j* (ξ_{ij}=1, otherwise ξ_{ij} = 0). $P(X_{ij}=1 \mid \underline{\alpha}, s, g)$ is the probability of answering item *j* correctly given that examinee *i* has a skill mastery vector of $\underline{\alpha}$ and error probabilities of s and g. The parameter s_j, denoting $P(X_{ij}=0 \mid \xi_{ij}=1)$, is the probability of answering incorrectly item *j* even though examinee *i* has all the attributes (skills) for the item (ξ_{ij} = 1). On the contrary, the parameter, g_j, representing $P(X_{ij}=1 \mid \xi_{ij}=0)$, is the probability of answering correctly item *j* even though examinee *i* has NOT mastered all the attributes for the item *j* (ξ_{ij} = 0). The false negative (s_j) and false positive (g_j) probabilities correspond to the positivity treated as a source of response variation in the unified model and the RUM, and g_j corresponds to r_{ik} in those two models. s_j and g_j can reflect on "examinees' slips and guesses, poor wording of the task description, inadequate specification of the Q matrix, use of an alternative solution strategy by the examinee, and general lack of model fit" (Junker & Sijtsma, 2001, p. 263).

The vector of $\xi_{ij} = (\xi_{i1}\ \xi_{i2}, \ldots, \xi_{ij})$ represents *ideal response patterns* in the rule space methodology's terms and is regarded as a Deterministic Input from each

examinee's skills mastery patterns (Junker & Sijtsma, 2001). Each ξ_{ij} plays as an "And" gate in Equation 2.3 because selecting between the probabilities (1-s_j) and g_j depends on the binary value of ξ_{ij} (0 or 1). 1-s_j = $P(X_{ij}=1|\xi_{ij}=1)$ is the probability of getting a correct answer for item *j*, given examinee *i* has all the attributes (skills) for the item, which corresponds to π_{ik} in the unified model and the RUM. 1-s_j will be selected only if ξ_{ij} is 1, otherwise, g_j will be chosen in the item response function. Then, because each X_{ij} is considered a Noisy observation of each ξ_{ij}, this model was referred to as Deterministic Inputs, Noisy "And" gate model.

2.4.2 NIDA Model

The item response function of the NIDA model is shown as

$$P(X_{ij}=1|\underline{\alpha},s,g) = \prod_{k=1}^{K} P(\eta_{ijk}=1|\alpha_{ik},Q_{jk}). \quad (2.4)$$

In the NIDA model, the latent variable η_{ijk} is newly introduced and indicates whether examinee *i*'s performance in the context of item *j* is consistent with possessing skill *k* (1 = consistent, 0 = inconsistent). The observed item response X_{ij} can be defined as X_{ij} = $\prod_{k=1}^{K}\eta_{ijk}$ .The two error probabilities were retained from the DINA model in the view point of skill level rather than in the item level. The false negative probability is expressed as

$$s_k = P(\eta_{ijk}=0|\alpha_{ik}=1,Q_{jk}=1), \quad (2.5)$$

which represents the probability that the examinee does NOT show consistent performance with having skill *k* required to solve item *j*. The false positive probability is expressed as

$$g_k = P(\eta_{ijk} = 1 \mid \alpha_{ik} = 0, Q_{jk} = 1), \tag{2.6}$$

which denotes the probability that the examinee's performance in the context of item *j* is consistent WITHOUT possessing skill *k* required to solve item *j*. Also, for the skills irrelevant to item *j*, the probability of getting $\eta_{ijk} = 1$ is fixed to 1 regardless of the value *a* (1 or 0) of α_{ik} in order not to influence the whole model,

$$P(\eta_{ijk} = 1 \mid \alpha_{ik} = a, Q_{jk} = 0) \equiv 1. \tag{2.7}$$

Finally, Equation 2.4 can be converted as follows:

$$P(X_{ij} = 1 \mid \underline{\alpha}, s, g) = \prod_{k=1}^{K} P(\eta_{ijk} = 1 \mid \alpha_{ik}, Q_{jk}) = \prod_{k=1}^{K} [(1 - s_k)^{\alpha_{ik}} g_k^{1-\alpha_{ik}}]^{Q_{jk}}$$

$$= \prod_{k=1}^{K} \left(\frac{1 - s_k}{g_k}\right)^{\alpha_{ik} Q_{jk}} \prod_{k=1}^{K} g_k^{Q_{jk}} . \tag{2.8}$$

The NIDA model is so named because "Noisy Inputs η_{ijk}, reflecting attributes α_{ik} in examinees, are combined in a Deterministic And gate X_{ij}" (Junker & Sijtsma, 2001, p. 265).

2.4.3 DINO Model

The item response function of DINO model is same as the one of DINA, as shown in Equation 2.3, except it is compensatory (the "Or" in DINO) instead of being conjunctive (the "And" in DINA).

$$P(X_{ij} = 1 \mid \underline{\alpha}, s, g) = (1 - s_j)^{\omega_{ij}} g_j^{1-\omega_{ij}}, \tag{2.9}$$

where $\omega_{ij} = 1 - \prod_{k=1}^{K} (1 - \alpha_{ik})^{Qjk}$, indicating if examinee *i* has satisfied at least one criterion (e.g., symptom) that Q-matrix includes for item *j* (ω_{ij}=1) or if examinee *i* has NOT satisfied any criteria that Q-matrix includes for item *j* (ω_{ij}=0). As in the DINA model,

s_j and g_j denote false negative and false negative probabilities, respectively. As mentioned above, DINO is a compensatory model which can be applied for medical and psychological disorder diagnosis. With the DINO model, it does not matter how many or which particular criteria have been satisfied if an examinee has satisfied one or more in the item (DiBello et al., 2007).

2.5 LLTM

Although a variety of latent class models successfully diagnosed an examinee's skill profile, they did not seem to incorporate cognitive models into test design except AHM. However, most latent trait models can link a cognitive theory to the test design and, in turn, can test the cognitive theory by the significance of the estimated item parameters in the model. Q-matrix is also applicable to latent trait models. The linear logistic test model (LLTM) was the first psychometric model that could effectively bridge cognitive psychology and item design. That is, the model can incorporate item stimulus features into the prediction of item success. LLTM is a generalization of the Rasch model which is given as

$$P(X_{is}=1)=\frac{\exp(\theta_s-\beta_i)}{1+\exp(\theta_s-\beta_i)}, \tag{2.10}$$

where $P(X_{is}=1)$ is the probability that person *s* passes item *i*, θ_s is the trait level of person *s*, and β_i is the difficulty parameter of item *i*. Because β_i does not include the cognitive variables (item stimulus features) involved in the item, it is replaced with a linear function of cognitive variables in LLTM as follows:

$$\beta_i = \sum_{k=0}^{K} \eta_k q_{ik} = \eta_0 q_{i0} + \eta_1 q_{i1} + \eta_2 q_{i2} + \ldots + \eta_k q_{ik}\,, \tag{2.11}$$

where η_k represents the effect of stimulus feature *k*, q_{ik} is the score (e.g., 0 = absence, 1 = presence) of stimulus feature *k* of item *i*, and $\eta_0 q_0$ is the intercept of the equation. The full LLTM combines Equation 2.10 with Equation 2.11;

$$P(X_{is} = 1) = \frac{\exp(\theta_s - \sum_{k=0}^{K} \eta_k q_{ik})}{1 + \exp(\theta_s - \sum_{k=0}^{K} \eta_k q_{ik})}\,. \tag{2.12}$$

The ***Q*** matrix which consists of q_{ik} (the indicators of *k* stimulus features of *i* items) is usually structured as a (*I*, *K*) matrix with rank *K*, where $K < I$. The method of coding the indicators, q_{ik}, can be dummy coding or scores for the item on cognitive model variables, depending on the purpose of the application. LLTM can be graphically represented as in Figure 2.2.

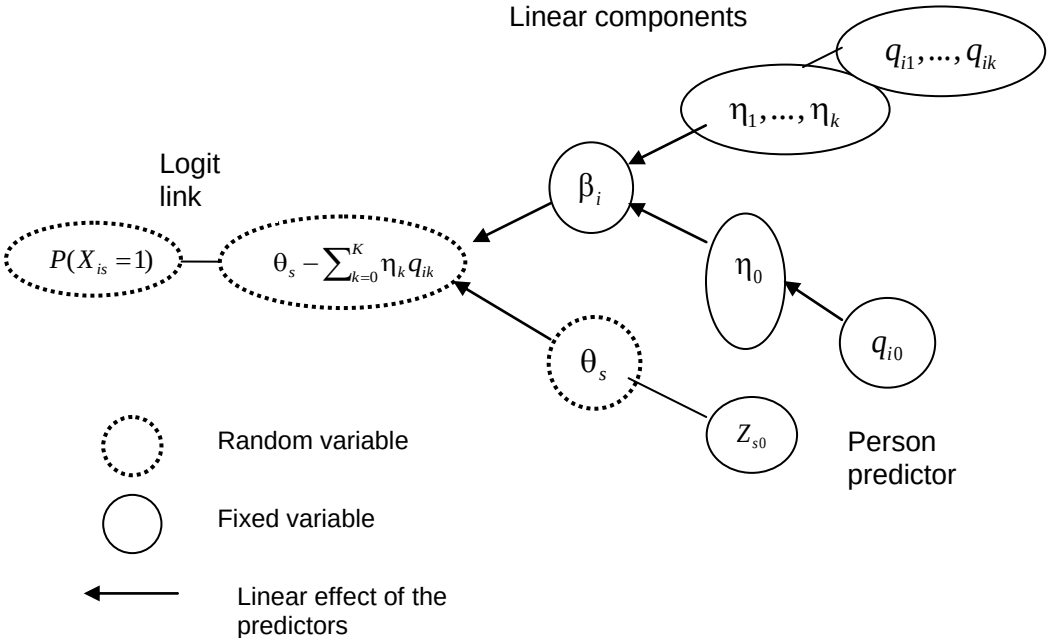


Figure 2.2 Graphical Representation of the LLTM (Wilson & De Boeck, 2004).

In Figure 2.2, dotted circles or ellipses represent random variables and regular circles or ellipses symbolize fixed variables. The figure shows q_{ik}, η_k, all the constants and their predicted value (β_i) as fixed variables and explains θ_s as a random variable. Logit link indicates that $P(X_{is}=1)$ is the function of $\theta_s - \sum_{k=0}^{K} \eta_k q_{ik}$. The contribution of each item is explained by the linear components of item stimulus features (q_{ik}) and their fixed effects (η_k) and the person predictor (θ_s).

2.6 MLTM

Many ability and achievement test items require multiple skills or competencies to obtain a correct response. As shown in Figure 2.2, the SAT Algebra item required multiple skills (attributes) to arrive at a correct response such as comprehension of text, algebraic manipulation, linear functions, and simultaneous equations. Although MIRT-C models can be applied to identify multiple dimensions of an item, they are only appropriate for items in which the multiple skills are compensatory. One important aspect of multidimensionality is that the skills or processing stages in the items often are sequentially dependent. Typical cognitive models for tasks postulate a flow of information from one stage to another. Thus, the assumptions underlying the MIRT-C models that low trait levels on one skill or stage can be compensated for by high trait levels on other skills or stages do not fit well with the cognitive psychology views of task performance.

Leighton et al. (2004) identified four possible forms of the hierarchical structures of attributes, as shown in Figure 2.3. In all the structures, attribute 1 is considered

prerequisite to the other attributes that follow. Except in Figure 2.3 (D), the *unstructured hierarchy*, there are orderings among attributes and there are unique relationships between the total score and the expected examinee response pattern (Leighton, et al., 2004). The four possible hierarchical structures in Figure 2.3 can be expanded and combined to apply to more complex hierarchies, where the complexity varies with the cognitive problem-solving task. Therefore, a model that can reflect the sequential dependency among the processing stages is necessary to assess properly the source of multidimensionality in the item domain, which better explains why an examinee fails a specific item.

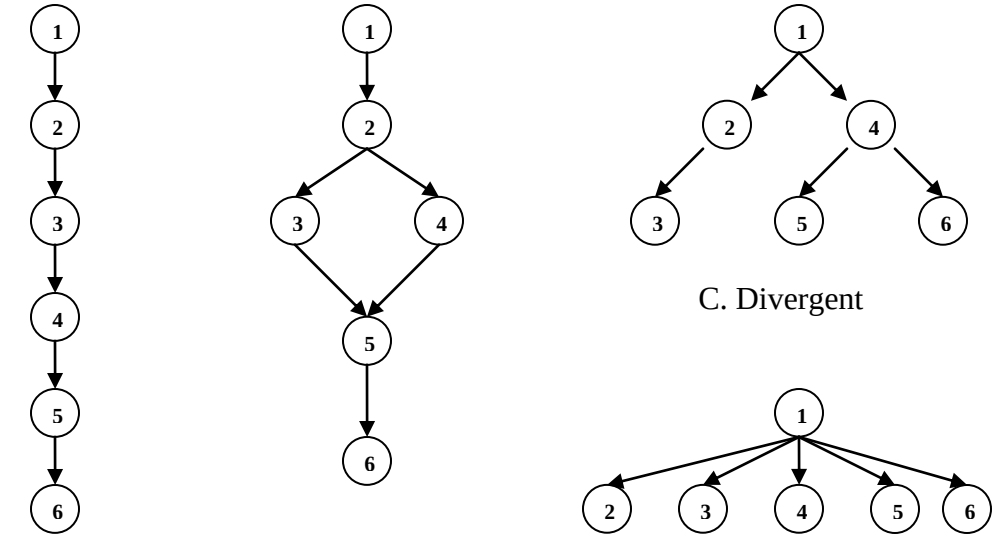


Figure 2.3 Four different hierarchical structures (Leighton et al., 2004).

The multidimensional latent trait model (MLTM; Whitely, 1980; Embretson, 1991; Embretson & Yang, 2006) is based on a continued product of processing outcome

probabilities to reflect the sequentially dependent stages (called component, here), as follows:

$$P(X_{is}=1)=\prod_k P(X_{isk})=\prod_k \frac{\exp(\theta_{sk}-\beta_{ik})}{1+\exp(\theta_{sk}-\beta_{ik})}, \quad (2.13)$$

where $P(X_{is}=1)$ is the probability of success for person *s* on item *i* and $\prod_k P(X_{isk})$ is the product of success on each processing component *k*, given the correct outcome of the preceding component. The right side of the equation contains Rasch models for the probability of success on each component, where θ_{sk} is the trait level of person *s* on component *k* and β_{ik} is the difficulty of item *i* on component *k*.

2.7 GLTM

The general component latent trait model (GLTM; Embretson, 1984) is the generalization of the MLTM that incorporates a mathematical model to relate the difficulty of each component (β_{ik}) to stimulus features in the item. For example, paragraph comprehension items, in which a short paragraph is followed by a question based on the paragraph, have two major components, text representation and decision. The difficulty of each component is related to stimulus features in the item. That is, text representation depends on vocabulary level and syntactic complexity while decision depends on the inference level and the amount of relevant text for the question (Gorin & Embretson, 2006). Therefore, β_{ik} is predicted by the weighted sum of underlying stimulus features as follows:

$$\beta_{ik}=\sum_{m=0}^{m}\eta_{km}q_{ikm}, \quad (2.14)$$

where q_{ikm} is the score of stimulus feature *m* on component *k* for item *i*, η_{ikm} is the weight of stimulus feature *m* on component *k*, and $\eta_{k0}q_{ik0}$ is an intercept. The full GLTM combines Equation 2.13 with Equation 2.14.

$$P(X_{isT}=1)=\prod_k\left[\frac{\exp(\theta_{sk}-\sum_{m=0}^{m}\eta_{km}q_{ikm})}{1+\exp(\theta_{sk}-\sum_{m=0}^{m}\eta_{km}q_{ikm})}\right]. \quad (2.15)$$

The GLTM enables an examination of how the underlying stimulus features will impact the difficulty of each component (β_{ik}) based on pre-established cognitive theories. Since GLTM is an extension of the MLTM, it also estimates individual ability on each component (also called cognitive attribute) as a continuous variable, thus giving detailed information about an examinee's skill profile.

GLTM may be contrasted to the latent class model (e.g., RUM, DINA, NIDA, AHM). Both types of models require a Q-matrix that specifies the sources of complexity in the items. In the latent class models, each combination of attributes potentially can define a mastery class or state. However, the number of classes can become quite large, even when just a few attributes are scored on items. The large number of classes may result in too fine distinctions since most achievement tests typically fit unidimensional IRT models fairly well. GLTM, in contrast, provides estimates of a few major component trait levels for examinees. Component trait levels also have diagnostic potential because they have implications for the likelihood that a person solves items with specific combinations of attributes.

Like MLTM, GLTM can be estimated readily when both subtask and full task data are available for the same item. The GLTM can be estimated with full task data if the stimulus factors adequately describe the components (e.g., Embretson & McCollam,

2000). For example, in the Embretson (1995) study, a working memory capacity component was separated from control processes in performance on spatial ability items because a highly predictive model for the difficulty of working memory load was available. Other circumstances in which GLTM can be estimated include setting constraints, data augmentation, and component structures that vary between items.

Finally, a generalized version of MLTM (Embretson & Yang, 2006, 2007) is especially applicable to cognitive diagnosis. The model is appropriate for items in which the mixture of components required for solution is varied. For example, in mathematics items, such as those found on the GRE, some items require only procedural skills (i.e., problems that contain only equations), while others require integration but no computation. Embretson and Yang (2006, 2007) show how the generalized version of MLTM is estimated with no requirement of special item subtasks. A similar generalization of GLTM readily follows from Embretson and Yang (2006, 2007).

2.8 LCDM

Math tests are typical examples where mastery of attributes cannot make up for nonmastery of the other attributes which are required to solve an item (non-compensatory). Thus, non-compensatory models, more specifically conjunctive models such as fusion, DINA, MLTM, and GLTM, are especially useful for diagnosing mathematical skills. In contrast, disjunctive models, such as DINO, are useful for the skill diagnosis where mastery of a subset of the attributes is sufficient to get a high probability of solving the item (Henson, Templin, & Willse, 2009). The log-linear cognitive diagnostic model (LCDM; Henson et al., 2009) is a generalized model to express both

conjunctive and disjunctive models. In LCDM, a non-compensatory model is expressed as a model where the relationship between any attribute (e.g., A_1) required in an item and the item response (x) depend on mastery or nonmastery of the remaining required attribute (e.g., A_2) while a compensatory model is expressed as a model where the conditional relationship of A_1, A_2 and x does not exist (Henson et al., 2009). The general form of the LCDM is as follows:

$$P(X_{ri} = 1|\boldsymbol{\alpha_r} = \boldsymbol{\alpha_c}) = \frac{\exp\left[\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)\right]}{1+\exp\left[\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)\right]}, \tag{2.16}$$

where $P(X_{ri} = 1|\boldsymbol{\alpha_r} = \boldsymbol{\alpha_c})$ is the probability that respondent r with the attribute-mastery profile $\boldsymbol{\alpha_c}$ correctly responds to the i^{th} item. $\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$ can be rewritten as:

$$\begin{aligned}\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \textstyle\sum_{u=1}^{K} \lambda_{i,1,(u)}(\alpha_{ru} q_{iu}) \\ + \textstyle\sum_{u=1}^{K} \sum_{v>u} \lambda_{i,2,(u,v)}(\alpha_{ru}\alpha_{rv} q_{iu} q_{iv}) + \cdots,\end{aligned} \tag{2.17}$$

where,

- $\boldsymbol{\lambda}_i^T$ is a $1 \times (2^K - 1)$ vector of weights (k = # of attributes) for the i^{th} item.
  For example, $\lambda_{i,1,(1)}$ represents a simple main effect of attribute 1, $\lambda_{i,1,(2)}$ refers to a simple main effect of attribute 2, and $\lambda_{i,2,(1,2)}$ represents a two-way interaction of attributes 1 and 2. $\lambda_{i,0}$ is an intercept.
- $\boldsymbol{q}_i$ is the Q-matrix entries of attributes to be measured in the i^{th} item ($k \times 1$ vector).
- $\boldsymbol{\alpha}_r$ represents the attribute mastery profile of respondent r ($1\times k$ vector).
- $\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$ is a set of linear combinations of $\boldsymbol{q}_i$ and $\boldsymbol{\alpha}_r$.

Therefore, the probability of a correct response for an item which requires two attributes (A_1 and A_2) can be defined as:

$$P(X_{ri} = 1|\boldsymbol{\alpha_c}) = \frac{\exp\left[\lambda_{i,0} + \lambda_{i,1,(1)}(\alpha_1) + \lambda_{i,1,(2)}(\alpha_2) + \lambda_{i,2,(1,2)}(\alpha_1\alpha_2)\right]}{1+\exp\left[\lambda_{i,0} + \lambda_{i,1,(1)}(\alpha_1) + \lambda_{i,1,(2)}(\alpha_2) + \lambda_{i,2,(1,2)}(\alpha_1\alpha_2)\right]}. \tag{2.18}$$

In this equation, if attribute 1 (A_1) is mastered (α_1=1), then the probability of a correct response increases by a factor of $e^{\lambda_{i,1,(1)}}$ given that other attribute (attribute 2) has not been mastered. $\lambda_{i,2,(1,2)}$ represents the extent to which the conditional relationship of A_1 and the item response depends on attribute 2 (A_2). Thus, if A_2 is mastered (α_2=1), the probability of a correct response increases by a factor of $e^{\lambda_{i,1,(2)}+\lambda_{i,2,(1,2)}}$. Such a model in Equation 2.17 can be extended to include all possible main effects and interactions of attributes.

One of the contributions of LCDM is that this model can provide empirical information regarding the relationship between attribute mastery and the item response without having to specify a type of model such as compensatory or non-compensatory (Henson et al., 2009). In other words, LCDM can show what type of model could have better fit for some test items. However, as one attribute is added in the model, the number of item parameters of LCDM will be doubled. For example, there are two item parameters (including the intercept) for one attribute, four item parameters for two attributes, eight parameters for three attributes, sixteen parameters for four attributes, and thirty two parameters for five attributes. Such large number of item parameters (2^k) is a drawback in LCDM because it causes even heavier computational demand than those of fusion model (k + 2) and the DINO model (2k) as well as large standard errors for the parameter estimation.

CHAPTER 3

BAYESIAN STATISTICAL INFERENCE

3.1 Introduction

There are two main approaches to statistical inference. The first is classical approach (also called frequentist approach) which has been a mainstream of statistical inference so far. The second approach is named Bayesian approach. Although both approaches are based on probability, Gillies (2000) defined the frequentist view of probability as limiting frequency of an outcome in a long series of similar events and the Bayesian (also called subjective) view of probability as degree of belief of the event. In the classical approach, parameters- the numerical characteristics of the population- are considered fixed but unknown constants, thus confidence statements such as 95% or 99% confidence interval are used to make inference about the parameters. The confidence is determined by the average behavior of the procedure over all possible random samples. However, in the Bayesian approach, the unknown parameters are considered random variables, thus direct probability statements about the parameters can be made. This is more straightforward and useful for making inferences than the confidence statements in the classical approach (Bolstad, 2007). In a variety of fields of science including IRT parameter estimation (e.g., EAP, MMLE), Bayesian statistical inference has been a solution to overcome the limitation of the classical statistics. Currently, there is an upsurge in using Bayesian statistical methods for applied statistical analysis (Bolstad, 2007).

3.2 Bayes' Theorem

An English Presbyterian minister, the reverend Thomas Bayes (1702-1761), discovered Bayes' theorem, a single tool that Bayesian statistics relies on. His friend Richard Price found his paper about the theorem, *An Towards Solving a Problem in the Doctrine of Chances*, after his death and had it published in 1763 in *Philosophical Transactions of the Royal Society*. From the late 18th century, Bayesian approach to statistics had been extensively developed until the frequentist approach to statistics was developed in the 19th century and eventually came to dominate the field of statistics. Then, Bayesian approach had fallen from favor by the early 20th century, but it revived from the mid 20th century by De Finetti, Jeffreys, Savage, and Lindley, among others who completed current methods of Bayesian statistical inference (Bolstad, 2007).

In Bayes' theorem, inverse probability is used to find the predictive distribution of future observations based on prior knowledge and the information contained in the current observation. The mathematical statement of Bayes' theorem is given by

$$P(\theta_i|D) = \frac{P(D \cap \theta_i)}{P(D)} = \frac{P(\theta_i) \times P(D|\theta_i)}{\sum_i P(\theta_i) \times P(D|\theta_i)} \tag{3.1}$$

where θ_i is an unobservable event (parameter),

D is an observable event (data),

$P(\theta_i)$ is the prior probability of event θ_i,

$P(D|\theta_i)$ is the likelihood (conditional probability) of D given θ_i,

$\sum_i P(\theta_i) \times P(D|\theta_i)$ is the marginal probability of D, and

$P(\theta_i|D)$ is the posterior probability of θ_i given D.

For example, suppose we have an educational process and we desire to estimate the probability that the process is either in Good Condition (G) or Bad Condition (B). If we

observed a *success* in the first trial period, then, the probability of *G* given the observation of success, *P*(*G*|*success*), and the probability of *B* given the observation of success, *P*(*B*|*success*), are as follows, respectively:

$$P(G|success) = \frac{P(G) \times P(success|G)}{P(G) \times P(success|G) + P(B) \times P(success|B)} \text{ and}$$

$$P(B|success) = \frac{P(B) \times P(success|B)}{P(B) \times P(success|B) + P(G) \times P(success|G)}.$$

If priors and likelihoods are available from the historical data as follows, the posterior probabilities can be computed as

$$P(G|success) = \frac{.90 \times .95}{.90 \times .95 + .10 \times .70} = .924 \text{ and}$$

$$P(B|success) = \frac{.10 \times .70}{.10 \times .70 + .90 \times .95} = .076, \text{ respectively,}$$

where $P(G) = .90$, $P(success \mid G) = .95$, $P(success \mid B) = .70$,

$P(B) = .10$, $P(failure \mid G) = .05$, and $P(failure \mid B) = .30$.

Consecutively, suppose we observed failure in the second trial period. Then, the probability of *G* given the observation of failure, *P*(*G*|*failure*), can be computed as

$$P(G|failure) = \frac{P(G) \times P(failure|G)}{P(G) \times P(failure|G) + P(B) \times P(failure|B)}$$

$$= \frac{.924 \times .05}{.924 \times .05 + .076 \times .30} = .67.$$

Note that the priors, *P*(*G*) and *P*(*B*), are now .924 and .076, respectively, which were the posteriors in the previous stage, *P*(*G*|*success*) and *P*(*B*|*success*), respectively. That is, the posterior after the previous step is used as the prior for the next step in Bayes' theorem. Likewise, the probability of *B* given the observation of failure, *P(B|failure),* can be obtained by

$$P(B|failure) = \frac{P(B) \times P(failure|B)}{P(B) \times P(failure|B) + P(G) \times P(failure|G)}$$

$$= \frac{.076 \times .30}{.076 \times .30 + .924 \times .05} = .33$$

Notice that our belief about parameter (prior) is revised by the sample data which depends on the parameter in the Bayesian inference. Therefore, Bayesian approach offers following advantages over the classical approach to statistics:

First, it allows direct probability statements about the parameters based on the actual occurring data. This contrasts with the classical approach where inference probabilities about the unknown parameters are based on all possible data sets that may or may not occur. In other words, the Bayesian approach provides a way to estimate the probability of a hypothesis given data, $P(H_0|\text{data})$, while the classical approach is to estimates the probability of data given a hypothesis, $P(\text{data}|H_0)$. This is a very compelling reason to use Bayesian statistics because the direct probability statements about the parameters are more useful for statistical inferences (Bolstad, 2007).

Second, in the classical approach to statistics, any prior knowledge about the parameters is disregarded for the purpose of "objectivity" (p. xxi, Bolstad, 2007). However, it would be the waste of information if we just throw away prior knowledge about the parameters which can be obtained from previous study results or researcher's belief. Bayesian approach uses both sources of information (the prior information and the data) to find a posterior distribution of the parameter. Additionally, the posterior distribution obtained in the previous stage is used as a prior distribution in the subsequent statistical procedure. In other words, posterior distribution gives the relative weights to

each parameter value after analyzing the data, thus the Bayesian estimator has often smaller estimation error than the unbiased estimator in the classical approach.

Third, Bayesian statistics has a single tool, Bayes' theorem, which is easy to understand and can be used for all situations. On the contrary, classical statistics has many different tools and methods which are generally complex to understand and should be applied to different situations (Bolstad, 2007).

3.3 Bayesian Inference for Binomial Proportion (BIBP)

If a random experiment has the outcome of one of two mutually exclusive and exhaustive ways (e.g., success/failure), it is called *Bernoulli experiment*. If a Bernoulli experiment occurs several independent times so that the probability of success remains the same over the times, it is called *Bernoulli trials* (Hogg & Craig, 1995). The *Binomial* (n, π) distribution models the data from n Bernoulli trials with the probability of "success" of π. The possible observed number of "successes" (=y) will be 0, 1, 2,…,n in the data. If we hold y fixed at a certain number and let π vary over its possible values, the conditional probability (likelihood) of observation y given the parameter π is as follows:

$$f(y \mid \pi) = \binom{n}{y} \pi^{y} (1-\pi)^{n-y} \text{ for } 0 \le \pi \le 1. \tag{3.2}$$

Binary item responses (i.e., correct/incorrect) can be a good example of binomial data. If we apply the binomial likelihood function to the binary item responses, then y will be the total number of corrects out of independent responses to n items. The unknown parameter π can be interpreted as the probability that an examinee possesses and correctly applies the attributes required to solve the items, assuming the items are measuring the same attribute(s). In other words, π can be regarded as the degree of belief

that a student has mastered the attributes measured in the test items. Estimation of π is the goal of CDA. Using Bayes' theorem, the probability statement about the unknown parameter can be made as the posterior distribution of π given observed y:

$$g(\pi \mid y) = \frac{g(\pi) \times f(y \mid \pi)}{\int_0^1 g(\pi) \times f(y \mid \pi) d\pi}. \tag{3.3}$$

Interestingly, if we use a beta distribution as the prior probability, $g(\pi)$, then the posterior probability, $g(\pi \mid y)$, can be easily obtained (Bolstad, 2007). Probability density function of *beta* (a, b) for π is given by

$$g(\pi; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1}(1-\pi)^{b-1} \text{ for } 0 \le \pi \le 1, \tag{3.4}$$

where $\Gamma(a)$, $\Gamma(b)$, and $\Gamma(a+b)$ are the Gamma functions and $\pi^{a-1}(1-\pi)^{b-1}$ determines the shape of the curve. Notice that the right portion of the beta density has a similar form of the binomial probability density (see Equation 3.2) which is a product of π to a power times (1-π) to another power. It should be noted that when we multiply a beta distribution (prior) by the binomial distribution (likelihood), we can just add the exponents of π and (1-π), respectively. In Bayes' theorem, we can ignore the constants in the prior $g(\pi)$ and likelihood $f(y|\pi)$ which are not the functions of the parameter π because multiplying either the prior or the likelihood by a constant does not affect the results of the posterior (Bolstad, 2007). Therefore, the posterior probability $g(\pi \mid y)$ can be easily obtained without having to go through the integration. This gives

$$g(\pi \mid y) = \frac{\Gamma(n+a+b)}{\Gamma(y+a)\Gamma(n-y+b)} \pi^{y+a-1}(1-\pi)^{n-y+b-1}. \tag{3.5}$$

Notice that the posterior is also a function of the *beta* (*a'*, *b'*) distribution with $a' = a + y$ and $b' = b + n - y$. In other words, the number of success (y) was added to a and the number of failure (n-y) was added to b. Using a beta prior allows getting a beta posterior by the simple updating rule "add successes to a, add failures to b". Therefore, when we have binomial observation, using a beta prior makes it particularly easy to get a posterior. Therefore Beta distribution is called the *conjugate* family for the binomial observation distribution (Bolstad, 2007).

The shapes of the *beta* (a, b) family are different depending on the values a and b. Some examples of beta distributions are shown in Figure 3.1.

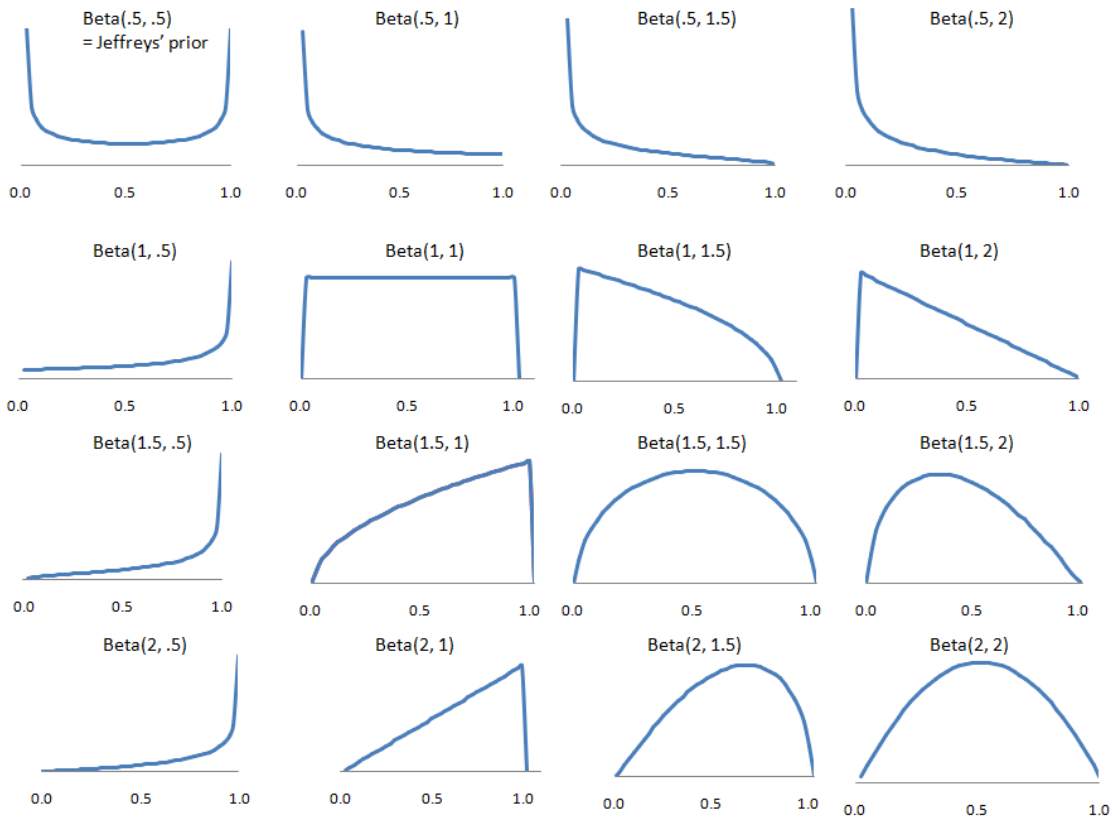


Figure 3.1 Samples of Beta Distribution (adapted from Bolstad, 2007)

The *uniform* (0,1) distribution is a special case of the beta distribution, *beta* (1, 1). When we have no knowledge or belief about a parameter before looking at the data, we often use the uniform prior which gives equal weight to all possible values of the parameter. Suppose we use the uniform prior, *beta* (1, 1), for the unknown π and observed one success from one trial (y = 1, n=1), then the posterior will be *beta* (2, 1) by adding y to a (=1). As shown in Figure 3.1, if $a > b$, the density has more weight in the upper half (> 0.5). In other words, the posterior mean of *beta* (2, 1) should be bigger than that of *beta* (1, 1) prior. The mean and variance of *beta* (a, b) distribution can be computed as follows:

$$\text{Mean: } E(\pi) = \int_0^1 \pi \times g(\pi; a, b)d\pi = \int_0^1 \pi \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\pi^{a-1}(1-\pi)^{b-1}d\pi \quad (3.6)$$

$$= \frac{a}{a+b} \text{ and}$$

$$\text{Variance: } Var(\pi) = E(\pi^2) - [E(\pi)]^2 = \frac{a(a+1)}{(a+b+1)(a+b)} - (\frac{a}{a+b})^2 \quad (3.7)$$

$$= \frac{ab}{(a+b)^2(a+b+1)} \ ,$$

respectively. Therefore, the means of uniform prior, *beta* (1, 1), and the posterior, *beta* (2, 1), will be 0.5 and 0.67, respectively.

3.4 Estimators for Proportion (π)

The posterior mean is considered the best estimate for a beta distribution, which has the smallest posterior mean square. In other words, it is closer to the parameter π on

average than any other estimates such as posterior mode and posterior median. Because π ranges between 0 and 1, the beta distribution does not have long tails (or large skewness), which makes the posterior mean a good measure of location for the beta distribution (Bolstad, 2007). However, the posterior mean is a biased estimator of π like the most other Bayesian estimators because, in Bayesian inference, parameters are considered unknown and random variables, thus unbiasedness is not emphasized by Bayesian statisticians (Bolstad, 2007). An estimator is unbiased if and only if

$$E(\hat{\pi}) = \int \hat{\pi} f(\hat{\pi}|\pi) d\hat{\pi} = \pi, \tag{3.8}$$

where $f(\hat{\pi}|\pi)$ is the sampling distribution of the estimator $\hat{\pi}$ given the parameter π. Bias is given by

$$bias(\hat{\pi}) = E(\hat{\pi}) - \pi. \tag{3.9}$$

Therefore, unbiased estimators which classical statistics emphasizes have zero bias while biased estimators have non-zero values for Equation 3.9. However, mean squared error (MSE) or root mean squared error (RMSE) is a more widely used criterion for judging estimators than the bias because MSE considers both the bias and the variance of the estimators. MSE is defined as (Bolstad, 2007)

$$MSE(\hat{\pi}) = E(\hat{\pi} - \pi)^2 = \int (\hat{\pi} - \pi)^2 f(\hat{\pi}|\pi) d\hat{\pi}$$

$$= bias(\hat{\pi})^2 + Var(\hat{\pi}). \tag{3.10}$$

Interestingly enough, Bolstad (2007) maintains that Bayesian's biased estimators often have smaller MSE values than the unbiased estimator in classical statistic. For example, suppose the parameter π is known to be 0.4 for ten binomial observations (n = 10). First, the unbiased estimator for π ($= \hat{\pi}_u$) and its variance are given by $\hat{\pi}_u = y/n$ and $Var(\hat{\pi}_u) = \pi(1-\pi)/n$, respectively, where y = the number of successes.

Then, MSE can be calculated as

$$MSE(\hat{\pi}_u) = bias(\hat{\pi})^2 + Var(\hat{\pi}) = 0^2 + \frac{\pi(1-\pi)}{n} = \frac{.4(.6)}{10} = 0.024.$$

Second, as the unbiased estimator for π (= $\hat{\pi}_b$), the posterior mean will be as follows if we use a uniform prior, *beta* (1,1):

$$\hat{\pi}_b = \frac{y+1}{n+2} = \frac{y}{n+2} + \frac{1}{n+2} = \frac{n\pi}{n+2} + \frac{1}{n+2}, \text{ for } y = n\pi.$$

The variance of $\hat{\pi}_b$ is given by $Var(\hat{\pi}_b) = [\frac{1}{n+2}]^2 \times n\pi(1-\pi)$.

Then, MSE can be computed as

$$MSE(\hat{\pi}_b) = bias(\hat{\pi}_b)^2 + Var(\hat{\pi}_b) = [E(\hat{\pi}_b) - \pi]^2 + Var(\hat{\pi}_b)$$

$$= \left[\frac{n\pi}{n+2} + \frac{1}{n+2} - \pi\right]^2 + \left[\frac{1}{n+2}\right]^2 \times n\pi(1-\pi)$$

$$= \left[\frac{(10)(0.4)+1}{12} - 0.4\right]^2 + \left[\frac{1}{12}\right]^2 \times 10 \times 0.4 \times 0.6 = 0.017.$$

MSE of the biased estimator $\hat{\pi}_b$ is smaller (= 0.017) than that of the unbiased estimator $\hat{\pi}_u$(= 0.024), which indicates that $\hat{\pi}_b$ is closer to the true value than $\hat{\pi}_u$ on average.

3.5 Bayesian Inference for Normal Mean

If we use a normal distribution as a prior in Bayes' theorem, it is also particularly easy to obtain a posterior given data without any numerical integration as in the beta distribution update. The posterior will also be a normal distribution where the parameters can be updated by a simple rule (Bolstad, 2007).

Normal distributions are symmetric, single-peaked, and bell-shaped, which are good descriptions for many distributions of real data. Also, many statistical inference

procedures based on normal distributions perform well for other roughly symmetric distributions. In the Bayesian framework, it is appropriate to consider normal distributions as likelihoods in many cases (Spiegelhalter, Abrams, & Myles, 2004). Suppose we have a random sample $y_1, y_2, \ldots, y_n$ taken from a normal distribution with mean μ and variance σ^2, where σ^2 is assumed to be known. The conditional probability (likelihood) of observation *y* given the parameter μ is

$$f(y|\mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}. \quad (3.11)$$

Then, the conditional probability of the unknown parameter μ given *y* (=posterior) can be expressed by Bayes' theorem as follows:

$$g(\mu|y) = \frac{g(\mu)\times f(y\,|\,\mu)}{\int g(\mu)\times f(y\,|\,\mu)d\mu}. \quad (3.12)$$

where g(μ) is a normal prior distribution with mean *m* and variance s^2.

The shape of the likelihood $f(y|\mu)$ is proportional to $e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$, $f(y|\mu) \propto e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$, because the left portion of Equation 3.11 (i.e., $\frac{1}{\sqrt{2\pi}\sigma}$) is independent of the parameter μ and multiplying the likelihood by any constant will cancel out in the posterior. Likewise, the shape of the prior will be as follows:

$$g(\mu) \propto e^{-\frac{1}{2s^2}(\mu-m)^2}.$$

Then, the shape of prior × likelihood will be given by (Bolstad, 2007):

$$g(\mu)\times f(y|\mu) \propto e^{-\frac{1}{2}\left[\frac{(\mu-m)^2}{s^2}+\frac{(y-\mu)^2}{\sigma^2}\right]} = e^{-\frac{1}{2\sigma^2 s^2/(\sigma^2+s^2)}\left[\mu-\frac{(\sigma^2 m+s^2 y)}{\sigma^2+s^2}\right]^2}.$$

In Bayes' theorem, it is always true that the posterior is proportional to the prior × likelihood because the denominator (the integration part) of Equation 3.12 is a constant:

$$g(\mu|y) \propto g(\mu) \times f(y|\mu) \propto e^{-\frac{1}{2\sigma^2 s^2/(\sigma^2+s^2)}\left[\mu-\frac{(\sigma^2 m+s^2 y)}{\sigma^2+s^2}\right]^2}. \quad (3.13)$$

Therefore, the posterior is a *normal* [*m'*, $(s')^2$] distribution having mean and variance given by

$$m' = \frac{(\sigma^2 m+s^2 y)}{\sigma^2+s^2} \; and \; (s')^2 = \frac{\sigma^2 s^2}{(\sigma^2+s^2)} \quad (3.14)$$

respectively. To sum up, using a *normal* (m, s^2) prior, we can obtain a *normal* [*m'*, $(s')^2$] posterior by the parameter updating rule in Equation 3.14 without having to go through the integration. Therefore, the normal distribution is called the conjugate family for the normal observation distribution (Bolstad, 2007).

The precision of a distribution (reciprocal of the variance) is given by

$$\frac{1}{(s')^2} = \frac{(\sigma^2+s^2)}{\sigma^2 s^2} = \frac{1}{s^2} + \frac{1}{\sigma^2} . \quad (3.15)$$

Notice that it is the prior precision ($1/s^2$) plus the observation (data) precision ($1/\sigma^2$). Then, the posterior mean in Equation 3.14 can be rewritten as

$$m' = \frac{\frac{1}{s^2}}{\frac{1}{\sigma^2}+\frac{1}{s^2}} \times m + \frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2}+\frac{1}{s^2}} \times y .$$

This is the weighted prior mean (m) by its precision ($1/s^2$) and the weighted observation (y) by its precision ($1/\sigma^2$). If we use a sample mean ($\bar{y}$) instead of a single observation value (y) and σ^2/n instead of σ^2 as the variance, then the posterior mean and variance will be given by

$$m' = \frac{\frac{1}{s^2}}{\frac{n}{\sigma^2}+\frac{1}{s^2}} \times m + \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{s^2}} \times \bar{y}\,, \tag{3.16}$$

$$(s')^2 = \frac{(\sigma^2/n)s^2}{(\sigma^2/n)+s^2} = \frac{\sigma^2 s^2}{\sigma^2+ns^2}\,, \tag{3.17}$$

where m and s^2 are the prior mean and variance, and

$\bar{y}$, σ^2/n, and n are the data (sample) mean, variance, and size.

This updating rule is also applicable for the flat priors such as uniform and Jeffreys' (Bolstad, 2007). Spiegelhalter et al. (2004) introduced simpler but equivalent formulae to those in Equation 3.16 and 3.17. They used the same variance σ^2 for both the prior and the data, but a sample size n_0 for the prior as follows:

$$m' = \frac{n_0\mu+n\bar{y}}{n_0+n}, \tag{3.18}$$

$$(s')^2 = \frac{\sigma^2}{n_0+n}\,, \tag{3.19}$$

where μ, σ^2/n_0, n_0 are the prior mean, variance, and 'implicit' sample size, and

$\bar{y}$, σ^2/n, and n are the data mean, variance, and size.

For example, suppose we have the prior mean (m) of 20 and the variance (s^2) of 6^2 and the data mean ($\bar{y}$) of 25 from 10 samples given $\sigma^2 = 1^2$. Then, the posterior mean (m') and the variance $(s')^2$ will be obtained using the updating rule in Equation 3.16 and 3.17 as follows:

$$m' = \frac{\frac{1}{6^2}}{\frac{10}{1^2}+\frac{1}{6^2}} \times 20 + \frac{\frac{10}{1^2}}{\frac{10}{1^2}+\frac{1}{6^2}} \times 25 = 24.99,$$

$$(s')^2 = \frac{(1^2)(6^2)}{1^2 + (10)(6^2)} = 0.10.$$

If we apply Equation 3.18 and 3.19 to this example, m' and $(s')^2$ will be computed as

$$m' = \frac{(1/36)(20)+(10)(25)}{(1/36)+10} = 24.99,\ (s')^2 = \frac{1^2}{(1/36)+10} = 0.10,$$

where the implicit sample size of prior n_0 = 1/36 since $\frac{\sigma^2(=1^2)}{n_0} = 6^2$.

The example above shows that the two different versions of normal distribution updating formulae (Equation 3.16 ~3.17 and Equation 3.18 ~ 3.19) are equivalent. The prior, data, and the posterior distributions of the example are shown in Figure 3.2. As shown in the figure, the posterior distribution has the smallest variance, thus providing more accurate measure of the parameter. This is why the Bayesian estimators have often smaller MSEs than the unbiased estimators of classical statistics.

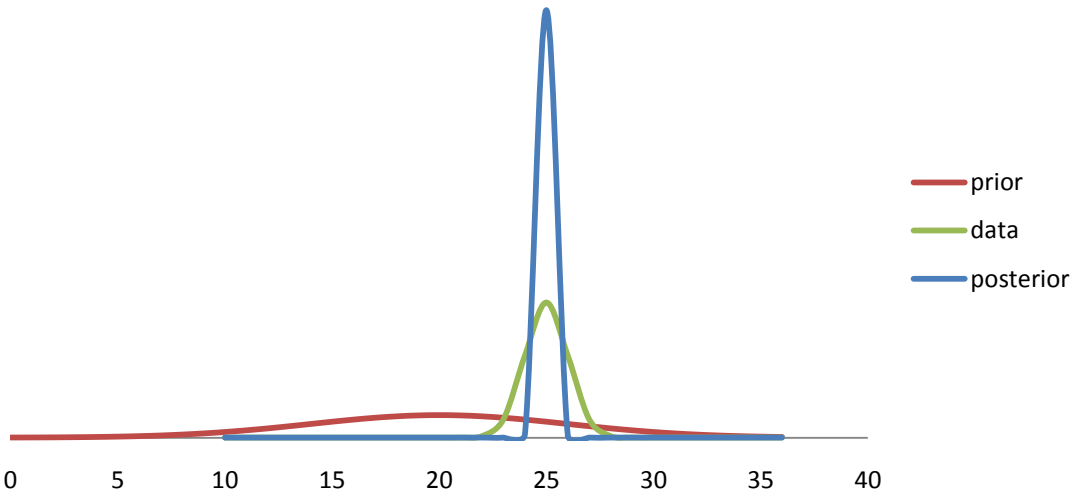


Figure 3.2 Prior, Data, Posterior Distributions

CHAPTER 4

THE CURRENT STUDIES

4.1 Study Design/Purpose

Test items are developed for measuring specific attributes; number of the attributes that each item involves is various. Items with a simple item-attribute structure measure one or two attributes only while complex items measure several attributes together. Similarly, some attributes can be measured individually but some should be jointly measured with others due to the dependency of the attributes. In general, the total number of attributes to be measured in a test affects the complexity of the item-attribute structure. The most effective way of applying BIBP to diagnostic assessment may differ depending on the test item-attribute design as well as the attribute-mastery patterns of examinees. In the current studies, effective ways of applying the BIBP method were explored using real data studies and simulation studies. Also, the diagnosis results of other preexisting diagnosis models were compared to the result of BIBP for the validity check.

Three studies were conducted using a middle school mathematical achievement data. In Real Data Study 1 and Study 2, the BIBP method was applied to a data which involved different total number of attributes based on the mathematical standards or benchmarks; four attributes (based on four mathematical standards) or ten attributes (based on ten benchmarks of the four standards), respectively. In Real Data Study 3, the BIBP method was compared to other two existing popular diagnosis models, DINA and LCDM, in diagnosing examinees' attribute mastery using the same data. For the

simulation study, two studies were conducted. In Simulation Study 1, the general accuracy of the BIBP method in the parameter estimation was evaluated. In Simulation Study 2, the impact of various conditions regarding attribute characteristics (e.g., correlation, difficulty) was examined for the accuracy of parameter estimation. Using same simulation data, the BIBP method was compared with DINA model in the parameter estimation accuracy on the various conditions.

4.2 Assumptions

There are three assumptions for BIBP method for CDA. First, it is assumed that the combined attributes are not necessarily same as simply adding the individual attributes. For example,

- $A_{12} \neq A_1 + A_2$,
- $A_{124} \neq A_{12} + A_4 \neq A_1 + A_2 + A_4$,

where A_1 = attribute 1, A_2= attribute 2, A_4= attribute 4, A_{12}= combined attribute of A_1 & A_2 (jointly measured in an item), A_{124}= combined attribute of A_1, A_2 & A_4 Bloom (1956) defined the *synthesis*- cognitive level (see Figure 4.1) as putting parts together to form a new whole. Forming a new whole does not mean simply adding two or more parts but creating a new meaning or structure. Therefore, the mastery probabilities of the combined attributes (e.g., π_{12}, π_{34}, π_{123}, π_{1234}) were separately estimated in this study in addition to the mastery probabilities of the single attributes (e.g., π_1, π_2, π_3, π_4). This is one of the unique features of the BIBP method unlike the other diagnostic models that estimate the mastery probabilities of only single attributes. Second, all the responses to *n* items which measure same attribute are independent (local independence for each

attribute). It other words, if there are three items (item #1, item #2, and item #3) which measure a same attribute (A_1), then the probability of answering correctly item #1 does not affect the probabilities of correct answer for item #2 or #3.

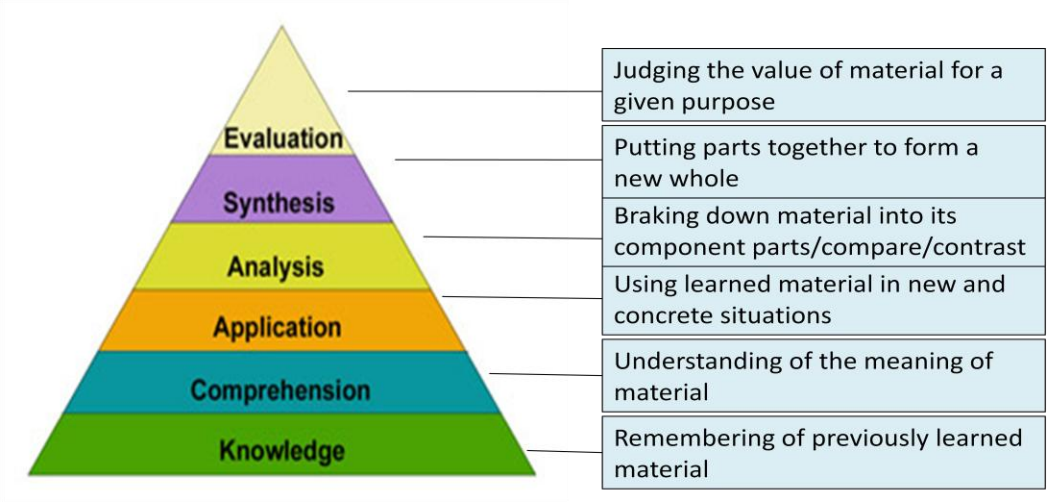


Figure 4.1 Bloom's Taxonomy of Cognitive Domain

Third, the examinee mastery probability (π_k) for attribute *k* is constant over all the responses to *n* items measuring attribute *k*. The last two assumptions are the fundamental assumptions of binomial likelihood function. Since BIBP method uses the binomial probability function as the likelihood, these two assumptions should be met.

4.3 Defining "Mastery"

In order to decide examinees' attribute mastery patterns, the state of "Mastery" should be defined first in terms of the attribute mastery probability (π). Mastery can be defined as the state "when students demonstrate a thorough understanding of content as evidenced by doing something substantive with the content beyond merely echoing it." (p.12, Wormeli, 2006). Mastery should include not only possessing the attribute but also

correctly applying it to the related problems. Although pure declaration of mastery may not be possible, the letter grades combined with 100-point scale, A (90-100), B (80-90), C (70-80), D (60-70), and F (< 60), are often used for most instructional decisions. The letter grades are typically interpreted as follows (Wormeli, 2006):

- Proximal mastery: A (Excellent), B (Good), and C(Fair)
- Limited proficiency: D
- No proficiency : F

Empirically, if an examinee answers three out of four questions correctly, the beta posterior mean of π will be also 70% when using Jeffreys' prior; since *a* (# of corrects) = 3 and *b* (# of incorrect) = 1, the posterior distribution will be *beta* (3 + 0.5, 1+ 0.5), thus $E(\pi) = a'/(a'+b') = 3.5/(3.5+1.5) = 0.7$ by Equation 3.6. Given the posterior mean of π = 0.7, the probability of nonmastery will be 0.3 (= $1 - \pi$). Then, the binomial probability of getting three or four correct answers although the examinee actually does not mastered the attribute (i.e., false positive probability) will be as follows:

$$f(y \geq 3) = \binom{4}{3} 0.3^3 (1-0.3)^{4-3} + \binom{4}{4} 0.3^4 (1-0.3)^{4-4} = .0756 + .0081 = .0837$$

Therefore, we can assume with more than 90% confidence that the examinee masters the attribute if he/she answers three out of four items correctly. In the current studies, the cutoff π-value for deciding the mastery or nonmastery was set to 0.7. Thus, if the estimated posterior mean of π for an attribute was equal to or bigger than 0.7, then it was considered as the evidence of 'Mastery' of the attribute, otherwise 'Nonmastery.'

CHAPTER 5

REAL DATA STUDY 1: FOUR-ATTRIBUTE DATA

The goal of this study was to find the way to estimate examinees' mastery probabilities and to diagnose their attribute mastery patterns using the BIBP method when a test involves four attributes. As mentioned above, the posteriors of single-attribute parameters (e.g., π_1, π_2) as well as multiple-attributes parameters (e.g., π_{12}) were obtained. The Excel program was used for the parameter estimation in the BIBP method.

5.1 Method

5.1.1 Subjects and Instruments

A mathematical achievement test of 86 items was administered to 8^{th} grade students in a Midwestern state at the end of each school year. Binary item responses (correct/incorrect) from a random sample of 2,993 students were obtained. All items in the test were multiple-choice items with four options. The mathematical test items were developed by the blueprint which represents four mathematical standards areas: (1) Number and Computation, (2) Algebra, (3) Geometry, and (4) Data. There are several benchmarks within the standards (see Appendix A for detailed information about the benchmarks). Each of the mathematical items was originally designed to involve a single standard (e.g., A_1) based on the blueprint, but it was found that some items required the mastery of multiple standards (e.g., A_{12} or A_{124}). Thus, a Q matrix, a incidence matrix of the attributes involved in each item, was developed by a mathematician who was

experienced in state assessment. Table 5.1 presents the item design based on the Q matrix in which each of the 86 items measure a single or multiple attributes.

Table 5.1 *Item Design for the Four Attributes*

Block	*Attribute*				*Item #*	*Total number of items*	*Parameter*
	1	2	3	4			
1	1	0	0	0	6, 8~12, 53, 71~74	11	π_1
2	0	1	0	0	24~35, 83~86	16	π_2
3	0	0	1	0	19~23, 43~46, 75~82	17	π_3
4	0	0	0	1	36~42, 54~57	11	π_4
5	1	1	0	0	1, 2~5, 7, 48~52, 65~70,	17	π_{12}
6	0	1	1	0	47	1	π_{23}
7	1	0	0	1	14	1	π_{14}
8	1	1	0	1	13, 15~18	5	π_{124}
9	1	1	1	1	58~64	7	π_{1234}

Note: 1=measuring, 0 = not measuring

As shown in the table, 65 out of 86 items (76%) involved a single attribute (Blocks 1 through 4) while 21 items (23%) involved more than two attributes (Block 5 through 9). More than five items belong to each block except Blocks 6 and 7 that include only one item each. Only a single test item should not be good enough to measure any attribute effectively and this issue will be discussed later (in Step 2). In Table 5.1, π_1 refers to the mastery probability of A_1, π_{12} refers to the mastery probability of the combined attribute of A_1 and A_2 and so on. The four standards were used as four attributes to measure in this study.

5.1.2 Estimating Single-Attribute Parameters (Step 1)

The single-attribute parameters, π_1, π_2, π_3, and π_4, were estimated as the first step. In order to obtain the posterior of each of the four parameters, Jeffrey's prior, *beta* (0.5, 0.5), and the responses to the items of blocks 1, 2, 3, and 4 were used. When we do not

have any prior knowledge about a parameter, generally the uniform prior is widely used, which gives equal weight to all possible values of the parameter. However, for diagnosing the attribute mastery pattern (mastery/nonmastery), Jeffrey's prior may be more appropriate to apply to the data because it puts more weight on the two extreme sides (0 and 1). In reality, the distribution of the attribute mastery probability is usually bimodal (mastery or nonmastery rather than the middle). Therefore, Jeffrey's prior was adopted as the priors of π_1, π_2, π_3, and π_4. Then, the posterior of each parameter was obtained by the simple updating rule "add successes to *a*, add failures to *b*" as elaborated in Chapter 3. For example, because block 1 includes eleven items which involves A_1 (see Table 5.1), the posterior of π_1 can be computed as *beta* (0.5 + *y*, 0.5 + 11 - *y*), where *y* = number of corrects in the eleven items. The same rule was applied to the posterior of π_2, π_3, and π_4, respectively. The posterior mean of each parameter was used as the parameter estimate (=$\hat{\pi}_k$) as mentioned earlier (see Chapter 3.4)

5.1.3 Estimating Multiple-Attribute Parameters (Step 2)

The posteriors of the five multiple-attribute parameters (π_{12}, π_{23}, π_{14}, π_{124}, and π_{1234}) were estimated, as the second step, using the response data of block 5, 6, 7, 8, and 9, respectively. In this step, it should be noted that the posteriors of the single-attribute parameters estimated in Step 1 were used as the priors of the multiple-attribute parameters. For example, for the prior of π_{12}, the posterior of either π_1 or π_2 was used depending on the examinee's ability. That is, if an examinee's $\hat{\pi}_1$ is smaller than $\hat{\pi}_2$, then, the posterior of π_1 was used as the prior of π_{12} of the examinee because $\hat{\pi}_{12}$should be equal to or lower than $\hat{\pi}_1$. For example, suppose that the obtained posteriors of π_1 and π_2 of an examinee were a *beta* (9.5, 2.5) with *M* = 0.79 and *beta* (15.5, 1.5) with *M* = 0.91,

respectively. Then, the posterior of π_1, *beta* (9.5, 2.5), would be used as the prior of π_{12} of the examinee because $\hat{\pi}_1$(= 0.79) is smaller than $\hat{\pi}_2$ (= 0.91). Then, the posterior of π_{12} would be computed as *beta* (9.5 + *y*, 2.5 + 17 – *y*) based on the responses to the 17 items that involve A_{12} (block 5 in Table 5.1).

The same rule was applied to π_{23} and π_{14}. It should be noted that although each of π_{23} and π_{14} was involved by only one item as shown in Table 5.1, since its prior was obtained from the posterior of π_1, π_2, or π_3 which were involved by at least eleven items, the estimation error due to a single item was greatly reduced. This is one of the important advantages of using the Bayesian approach. As mentioned in Chapter 3.2, Bayesian approach uses both sources of information, the prior information and the data, to find a posterior distribution of the parameter, thus often resulting in smaller estimation error than the unbiased estimator in the classical approach. For π_{124}, the posterior of either π_{12} or π_4 was used as its prior although the posterior of π_{14} or π_2 could be also used for the prior of π_{124}. It is because the number of items involving π_{12} and π_4 is bigger than the number of items involving π_{14} and π_2, thus π_{12} and π_4 can give better prior information for π_{124} than π_{14} and π_2. Finally, for the mastery probability of all attributes combined (=π_{1234}), the posterior of either π_{124} or π_3 were selected as the prior of π_{1234}. Although other parameters' posteriors, such as single-attribute parameters or π_{12}, could be used as the prior, the combination of π_{124} and π_3 was chosen because especially the posterior of π_{124} was updated by many other parameters such as π_1, π_2, π_{12}, and π_4, thus providing a better source of prior information for π_{1234}. The procedure of estimating the posteriors of these nine parameters (π_1, π_2, π_3, π_4, π_{12}, π_{23}, π_{14}, π_{124}, and π_{1234}) is graphically displayed in Figure 5.1.

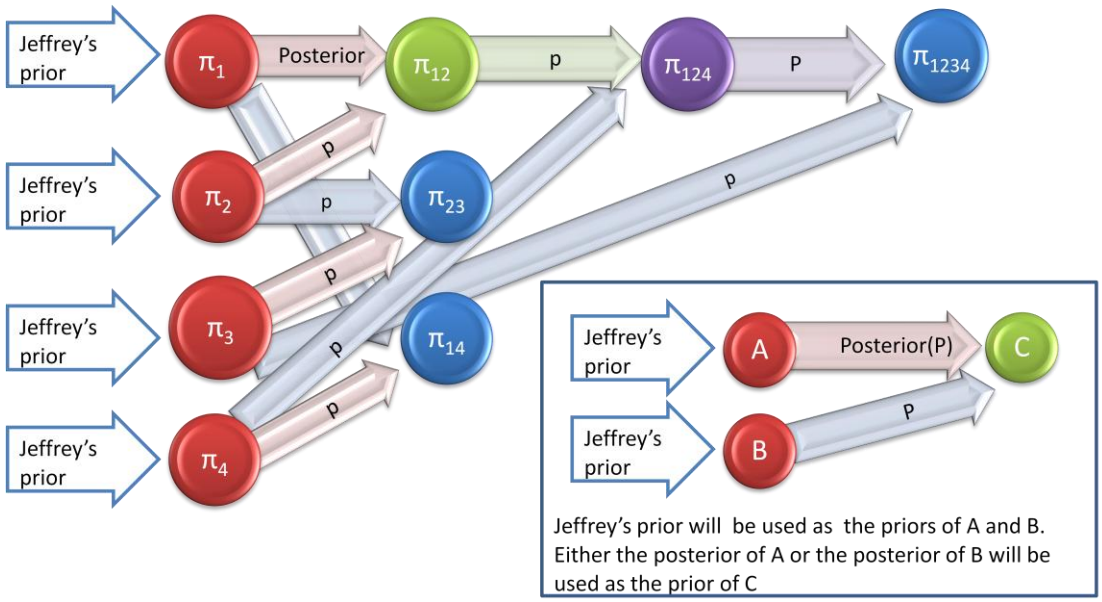


Figure 5.1 Estimating the Posteriors of the Attribute Mastery Probabilities in Study 1

5.1.4 Updating the Single-Attribute Parameters (Step 3)

In Step 1, the posteriors of the single-attribute parameters (π_1, π_2, π_3, & π_4) were estimated using the data of Blocks 1 through 4 only which directly measured the single attributes, thus the rest of data (from Blocks 5 through 9 in Table 5.1) which measured multiple attributes were not used. Based on the current item design in Table 4, it may not necessary to update the four single-attribute parameter estimates further because each of the single attributes was measured by a large enough number of items (at least eleven). However, this step was conducted as an effort to find a way to update the single-attribute parameter estimates using the response data to the multiple-attribute items.

In this step, the single-attribute parameter estimates were updated differently depending on examinees' ability levels on the attributes. For example, if an examinee showed the mastery of A_1 ($\hat{\pi}_1 \geq 0.7$) but nonmastery of A_2 ($\hat{\pi}_2 < 0.7$), then the responses

to items involving A_{12} (Block 5) depended on the mastery probability of A_2 (= π_2). In such case, the block 5-data was used to update the posterior of π_2 of the examinee using the *beta* updating rule: $a' = a + y$ and $b' = b + n - y$, where $n = 17$ since Block 5 includes 17 items. However, if both A_1 and A_2 were mastered ($\hat{\pi}_1 \geq 0.7$ and $\hat{\pi}_2 \geq 0.7$), the posteriors of both π_1 and π_2 were updated by the Block 5-data. Otherwise (if neither A_1 nor A_2 were mastered), no posterior of π_1 and π_2 could be updated because it was uncertain whether incorrect answers to block 5-items were due to the nonmastery of A_1 or nonmastery of A_2. Following is a proposed system of updating the posteriors of π_1, π_2, π_3, and π_4 using the response data from blocks 5 through 9.

- Using the Block 5-data (involving A_{12})
    - If $\hat{\pi}_1 < 0.7$ (nonmastery) and $\hat{\pi}_2 \geq 0.7$ (mastery), the posterior of π_1 will be updated
    - Else if $\hat{\pi}_1 \geq 0.7$ (mastery) and $\hat{\pi}_2 < 0.7$ (nonmastery), the posterior of π_2 will be updated
    - Else if both $\hat{\pi}_1$ and $\hat{\pi}_2 \geq 0.7$ (mastery), the posteriors of both π_1 and π_2 will be updated.
    - Otherwise, no updating for the posteriors of π_1 and π_2.
- Using the Block 6-data (involving A_{23})
    - Same rule as in block 5 will be applied to update the posterior of π_2 or (and) π_3.
- Using the Block 7-data (involving A_{14})
    - Same rule as in block 5 will be applied to update the posterior of π_1 or (and) π_4.

- Using the Block 8-data (involving A_{124})
    - If only one of $\hat{\pi}_1$, $\hat{\pi}_2$, and $\hat{\pi}_4 < 0.7$, then its posterior will be updated
    - Else if all $\hat{\pi}_1$, $\hat{\pi}_2$, and $\hat{\pi}_4 \geq 0.7$, the posteriors of all π_1, π_2, and π_4 will be updated.
    - Otherwise, no updating for the posteriors of π_1, π_2 and π_4.
- Using the Block 9-data (involving A_{1234})
    - If only one of $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{\pi}_3$ and $\hat{\pi}_4 < 0.7$, then its posterior will be updated by the data
    - If all of $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{\pi}_3$ and $\hat{\pi}_4 \geq 0.7$, then its posterior will be updated.
    - Otherwise, no updating for any of the posteriors of π_1, π_2, π_3, and π_4.

It was also evaluated whether this step would actually improve the accuracy of the parameter estimation using a simulation study in Chapter 8.

5.2 Result

5.2.1 Descriptive Statistics

The descriptive statistics of the raw score and attribute mastery probability estimate ($\hat{\pi}_k$) were provided in Table 5.2. Each descriptive statistic was reported for Step 1 & 2 and for Step 3, separately. Note that the raw scores and the multiple-attribute parameter estimates (e.g., $\hat{\pi}_{12}$, $\hat{\pi}_{124}$) remained same on both sides since Step 3 was conducted to update only the single-attribute parameters (e.g., π_1, π_2). The raw scores ranged from 22 to 86 (out of 86) with the *mean* of 60.68 (70.6%) and *SD* of 15.12. As shown in the table, the mean of $\hat{\pi}_3$ was highest (≈ .72) and the mean of $\hat{\pi}_{14}$ was lowest (≈ .60) in both Step 1& 2 and Step 3, which means that the examinees were more likely

to master A_3 (Geometry) and less likely to master A_{14} (combined attribute of 'Number and Computation' and 'Data') than any other attributes.

Table 5.2 *Descriptive Statistics of the Raw Score and Estimated Attribute Mastery Probability ($\hat{\pi}_k$) in Step 1 & 2 and Step 3 (N=2993)*

	Step 1 & 2				Step 3			
	Min.	*Max.*	*M*	*SD*	*Min.*	*Max.*	*M*	*SD*
Raw score	22	86	60.68 (71%)	15.12	22	86	60.68 (71%)	15.12
$\hat{\pi}_1$	0.0417	0.9583	0.6394	0.2124	0.0385	0.9881	0.6669	0.2125
$\hat{\pi}_2$	0.0882	0.9706	0.6702	0.1976	0.0882	0.9894	0.6817	0.1986
$\hat{\pi}_3$	0.0833	0.9722	0.7242	0.2142	0.0833	0.9808	0.7218	0.2116
$\hat{\pi}_4$	0.0417	0.9583	0.6394	0.1968	0.0417	0.9800	0.6633	0.2027
$\hat{\pi}_{12}$	0.1897	0.9828	0.6619	0.1761	0.1897	0.9828	0.6619	0.1761
$\hat{\pi}_{23}$	0.0833	0.9722	0.6305	0.2091	0.0833	0.9722	0.6305	0.2091
$\hat{\pi}_{14}$	0.0385	0.9615	0.5973	0.1925	0.0385	0.9615	0.5973	0.1925
$\hat{\pi}_{124}$	0.0294	0.9706	0.6003	0.1940	0.0294	0.9706	0.6003	0.1940
$\hat{\pi}_{1234}$	0.1000	0.9737	0.6209	0.1807	0.1000	0.9737	0.6209	0.1807

Table 5.3 presented the attribute difficulty, p_k (= proportion of all the examinees who have mastered attribute *k*). p_k of the nine attributes ranged from .32 to .57 in Step 1 and from .32 to .58 in Step 2. Note that the average p_k (= .43 in Step 1) was lower than the average percentage of the raw scores (71%). It was because the mastery of an attribute was defined as $\hat{\pi}_k \geq 0.7$ (see Chapter 4.3) in this study, thus examinees should answer at least three out of four items correctly in order to be classified as a mastery.

Like $\hat{\pi}_k$, p_k was highest for A_3 (.57 in Step 1&2 and .58 in Step 3) and lowest for A_{14} (= .32). In other words, about 60% of the examinees mastered A_3 while only 32% of the examinees mastered A_{14}. It should be also noted that $\hat{\pi}_k$and p_k of the single-attributes (A_1, A_2, A_3, A_4) were slightly increased in Step 3 after updating them by the multiple-attribute item response data.

Table 5.3 *Attribute Difficulty (p_k) in Step 1 & 2 and Step 3 (N=2993)*

Attribute	*Step 1 & 2*		*Step 3*	
	# of examinees who mastered	p_k	*# of examinees who mastered*	p_k
A_1	1419	0.47	1561	0.52
A_2	1477	0.49	1568	0.52
A_3	1694	0.57	1742	0.58
A_4	1314	0.44	1499	0.50
A_{12}	1369	0.46	1369	0.46
A_{23}	1230	0.41	1230	0.41
A_{14}	964	0.32	964	0.32
A_{124}	1000	0.33	1000	0.33
A_{1234}	1132	0.38	1132	0.38

Inter-attribute correlations were also presented in Table 5.4 and Table 5.5 for Step 1 & 2 and Step 3, respectively. The Pearson correlation of $\hat{\pi}_k$ with each other was considered an inter-attribute correlation. As shown in Table 5.4, the four single-attributes had medium inter-attribute correlations ranging from .614 to .722 while the multiple-attributes showed high inter-attribute correlations ranging from .703 to .925. Inter-correlations of the multiple attributes had relatively higher than those of the single attributes. It was observed that the inter-attribute correlations became higher in general when the two attributes shared more common attributes. For example, r ($\hat{\pi}_1$, $\hat{\pi}_{14}$) =.874 was bigger than r ($\hat{\pi}_1$, $\hat{\pi}_{23}$) = .709 and r ($\hat{\pi}_{14}$, $\hat{\pi}_{124}$) =.917 was even bigger than r ($\hat{\pi}_1$, $\hat{\pi}_{14}$) since A_1 and A_{14} had a common attribute A_1 and A_{14} and A_{124} shared more attributes (A_1 and A_4). As shown in Table 5.5, it was also an interesting finding that the inter-attribute correlations in Step 3 were generally higher than those in Step 1&2, which implied that updating the single-attribute parameter estimates in Step 3 increased their inter-attribute correlations.

Table 5.4 *Inter-Attribute Correlations (Step 1& 2)*

	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
$\hat{\pi}_1$	.672*	.687*	.614*	.833*	.709*	.874*	.832*	.806*
$\hat{\pi}_2$	1	.722*	.632*	.857*	.925*	.703*	.819*	.821*
$\hat{\pi}_3$		1	.637*	.785*	.872*	.710*	.757*	.807*
$\hat{\pi}_4$			1	.705*	.668*	.849*	.788*	.778*
$\hat{\pi}_{12}$				1	.868*	.834*	.888*	.891*
$\hat{\pi}_{23}$					1	.740*	.833*	.869*
$\hat{\pi}_{14}$						1	.917*	.887*
$\hat{\pi}_{124}$							1	.914*
$\hat{\pi}_{1234}$								1

* significant at the 0.001 level.

Table 5.5. *Inter-Attribute Correlations (Step 3)*

	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
$\hat{\pi}_1$	.817*	.746*	.751*	.884*	.782*	.873*	.863*	.842*
$\hat{\pi}_2$	1	.759*	.741*	.893*	.917*	.752*	.847*	.849*
$\hat{\pi}_3$		1	.701*	.789*	.873*	.713*	.760*	.814*
$\hat{\pi}_4$			1	.776*	.736*	.863*	.835*	.823*

* significant at the 0.001 level.

5.2.2 Individual diagnosis result

The BIBP method estimated the mastery probabilities ($\hat{\pi}_k$) and the mastery pattern (α_k) of each examinee for a total of nine attributes (A_1, A_2, A_3, A_4, A_{12}, A_{23}, A_{14}, A_{124}, and A_{1234}). Appendix B provided the parameter estimates for the first twenty and the last ten examinees in the Excel program view. Since examinees who have the same raw score may have a variety of different attribute-mastery patterns, the purpose of CDA is to provide the detailed attribute-mastery profile for each individual rather than just test score. Table 5.6 presents the diagnosis results for three examinees whose raw scores were 64 out of 86: $\hat{\pi}_k$ and α_k of the nine attributes estimated in Step 1&2 and Step 3.

Table 5.6 *Diagnosis Results for the Three Examinees (Raw Score: 64/86) by Step 1&2 and Step 3*

	Examinee ID	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
Step 1& 2	#116	**0.458**	0.912	0.806	0.375	0.603	0.816	0.423	0.559	0.605
	α_k	0	1	1	0	0	1	0	0	0
	#130	**0.208**	0.794	0.806	0.958	0.603	0.806	0.269	0.265	0.395
	α_k	0	1	1	1	0	1	0	0	0
	#2706	0.875	**0.500**	0.861	0.542	0.721	0.472	0.577	0.523	0.563
	α_k	1	0	1	0	1	0	0	0	0
Step 3	#116	**0.603**	0.917	0.816	0.375	0.603	0.816	0.423	0.559	0.605
	α_k	0	1	1	0	0	1	0	0	0
	#130	**0.607**	0.806	0.816	0.958	0.603	0.806	0.269	0.265	0.395
	α_k	0	1	1	1	0	1	0	0	0
	#2706	0.875	**0.700**	0.861	0.577	0.721	0.472	0.577	0.523	0.563
	α_k	1	1	1	0	1	0	0	0	0

Examinee #116 was diagnosed to have mastered A_2 and A_3 but not mastered A_1 and A_4 (α_k= 0110) for the four single attributes, while examinee #130 was diagnosed to have the mastery pattern of '0110' (α_k= 1, if $\hat{\pi}_k \geq .7$) in both Step 1& 2 and Step 3. Note that $\hat{\pi}_1$ was substantially increased for these two examinees in Step 3 as shown in Table 5.6. By examining their item response data, it was found that the two examinees performed poor on the 11 items involving A_1 (five and two corrects, respectively), but they did relatively well on the 17 items involving A_{12} (11 and 14 corrects, respectively). It was speculated that their $\hat{\pi}_{12}$ improved their $\hat{\pi}_1$ by the updating procedure in Step 3. The $\hat{\pi}_1$ change between Step 1 & 2 and Step 3 was graphically represented in Figure 5.2 (examinee #116) and 5.3 (examinee #130), respectively.

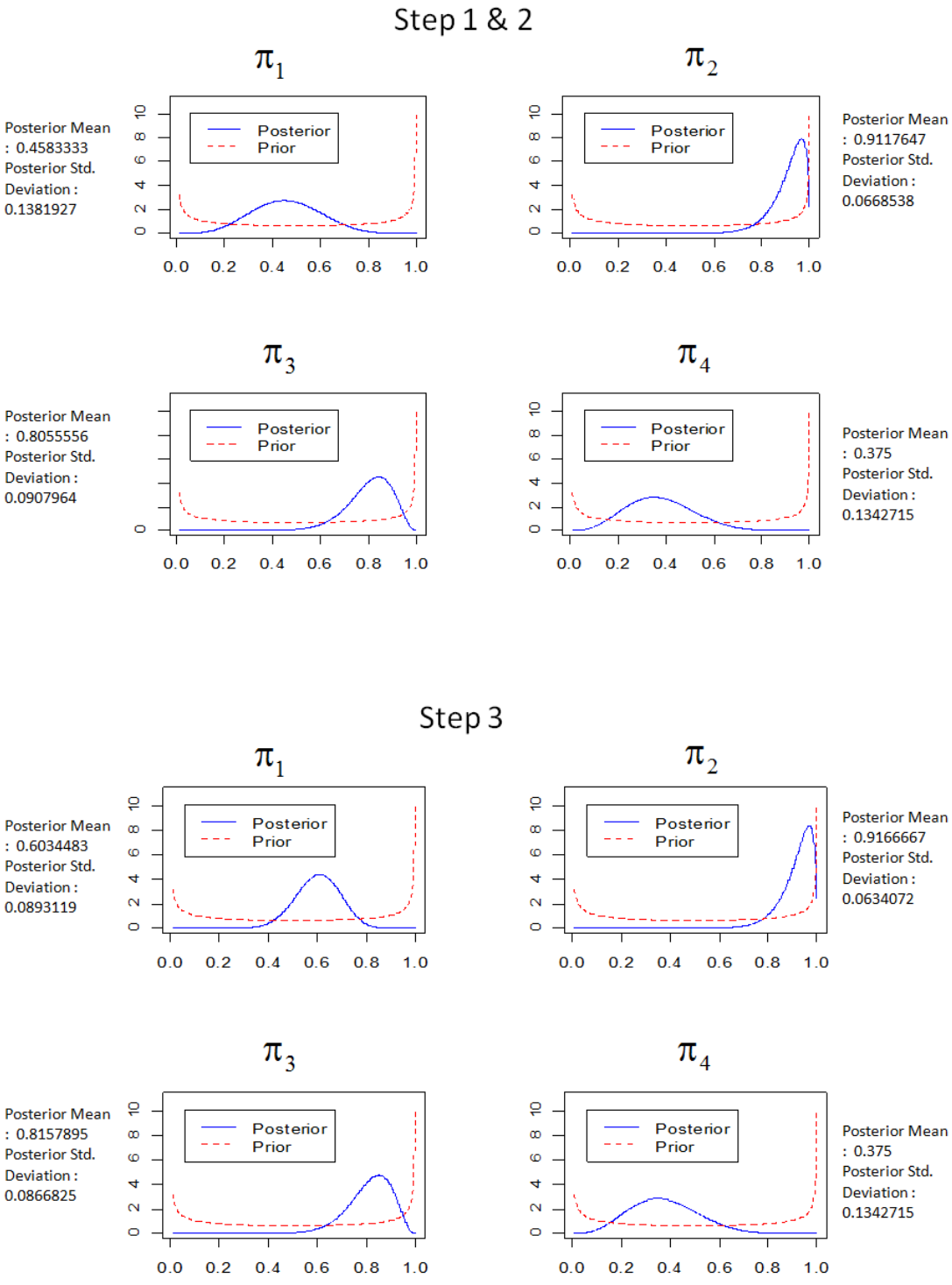


Figure 5.2 Estimated Posteriors of π_1, π_2, π_3, and π_4 for Examinee #116

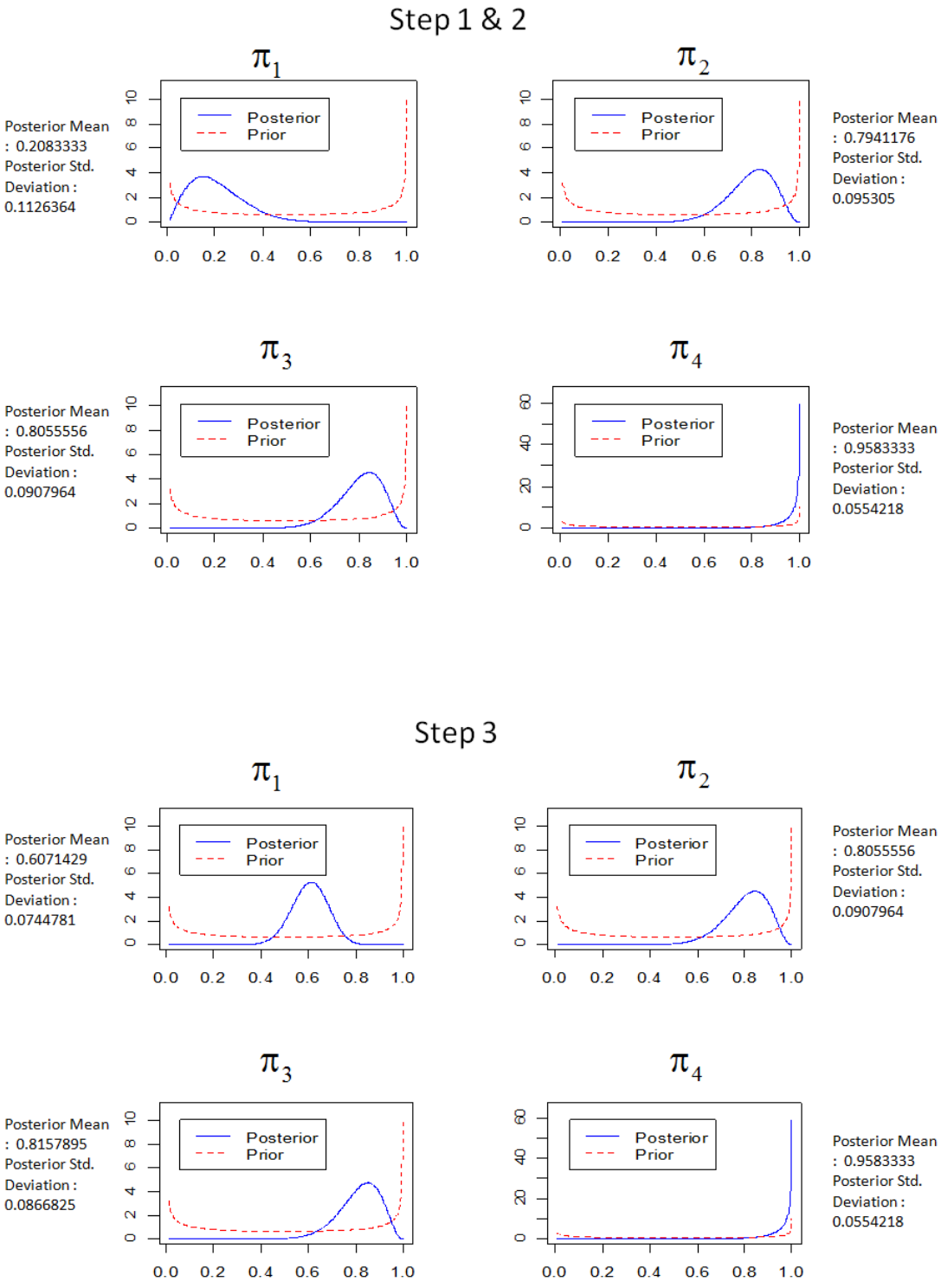


Figure 5.3 Estimated Posteriors of π_1, π_2, π_3, and π_4 for Examinee #130

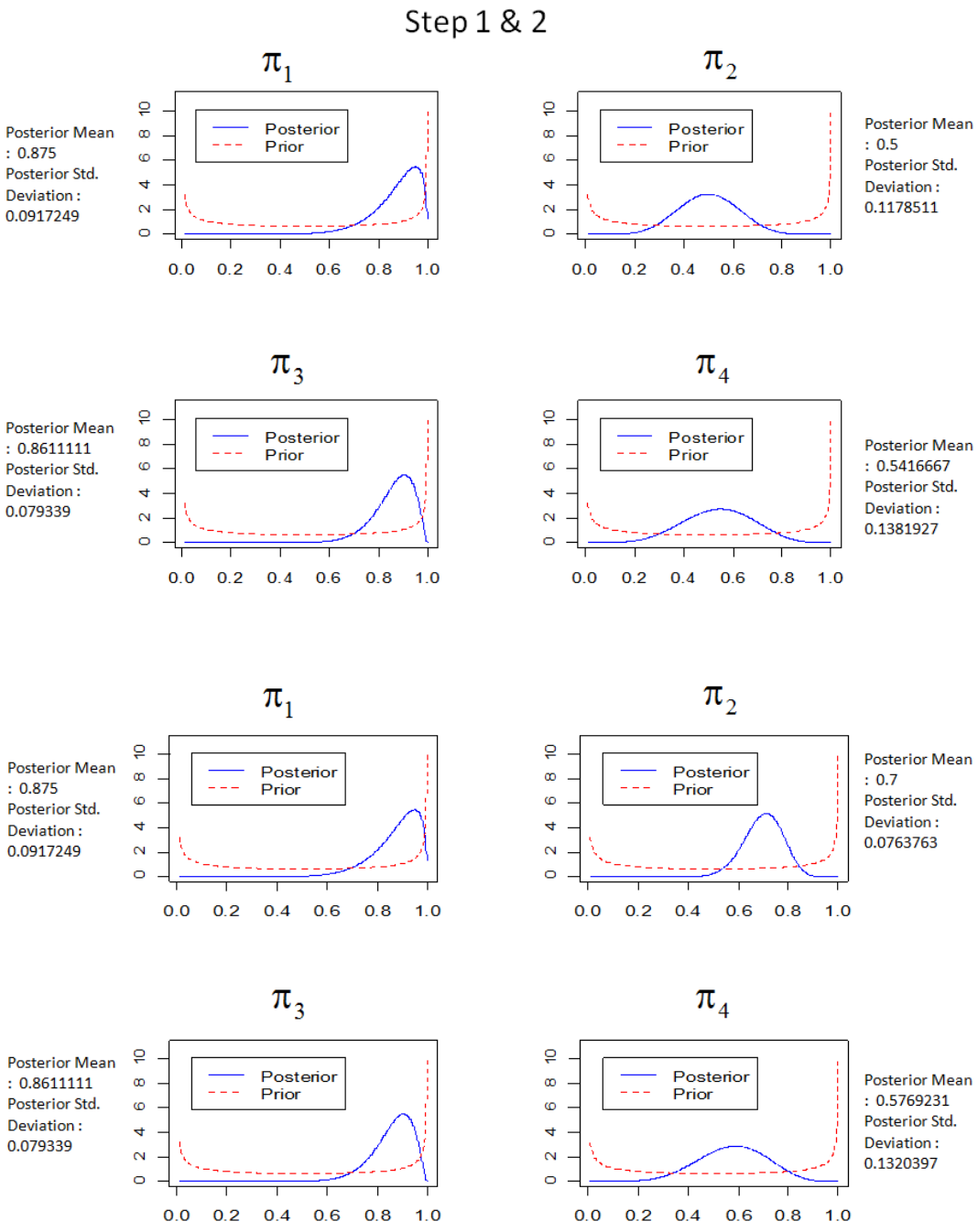


Figure 5.4 Estimated Posteriors of π_1, π_2, π_3, and π_4 for Examinee #2706

It was an interesting finding that Examinee #2706 showed the mastery pattern of '1010' in Step 1 & 2 (mastery of A_1, A_3) but '1110' in Step 3 (mastery of A_1, A_2, A_3). As

shown in Table 5.6, the value of $\hat{\pi}_2$ was changed from .50 to .70 in Step 3, thus this examinee was diagnosed to have mastered A_2 according to the decision rule of mastery (α_k= 1, if $\hat{\pi}_k \geq .7$). Similarly to examinees #116 and #130, extremely good performance on the items involving A_{12} (16 corrects out of 17) resulted in such a different diagnosis result on A_2 for this examinee. The changes of the parameter estimates for this examinee were graphically represented in Figure 5.4.

5.2.3 Discussion

The aberrant item response patterns on between the single-attribute (lower level) and the multiple-attribute (higher level) such as poor performance on A_1 or A_2, but better performance on A_{12} in the three examinees above may be due to several reasons.

Negative slips on the single-attribute items or positive slips (guessing) on the multiple-attribute items could result in the aberrant result. To detect the aberrant item response patterns of examinees, a variety of person-fit indices have been developed and applied. However, it is still unclear to decide whether the unusual response pattern is due to the negative slip or positive slip by the person-fit index only. Also, the aberrant response pattern may suggest that there is a compensatory nature between the attributes (e.g., A_1 or A_2) in a combined multiple attribute (e.g., A_{12}). Step 3 (updating the single-attribute $\hat{\pi}_k$ by multiple-attribute data) can be helpful for case of the negative-slips on single-attribute items but it will decrease the parameter estimation accuracy for the case of positive-slips on multiple-attributes items. It was examined whether the updating step generally improves the accuracy of parameter estimation in Simulation Study 1. Step 1&2 and Step 3 was compared in the accuracy of recovering true parameters.

CHAPTER 6

REAL DATA STUDY 2: TEN-ATTRIBUTE DATA

In this, BIBP method was applied to a more complex item design involving a total of ten attributes in a test, especially where every single attribute could not be measured directly by the test items. Therefore, it was explored how to infer π_k of such unmeasured single attributes from the parameter estimates of the attributes directly measured by the test items. The Excel program was used for all the parameter estimation.

6.1 Method

6.1.1 Subjects and Instruments

Subjects and instruments are the same as in Real Data Study 1. However, the ten benchmarks (see Appendix A) of the four mathematical standards were used as the ten attributes to be measured in this study. The first three benchmarks (1.Number Sense, 2.Number System and their Properties, 3.Computation) are the benchmarks of Standard 1 (Number and Computation). The next three benchmarks (4.Variabel, Equations, and Inequalities, 5. Functions, 6.Models) are of Standard 2 (Algebra). Then, Benchmark 7 (Geometric Figures and their Properties) and 8 (Geometry from an Algebraic Perspective) belong to Standard 3 (Geometry). Finally, Benchmark 9 (Probability) and 10 (Statistics) belong to Standard 4 (Data). The Q-matrix was also developed based on the ten benchmarks (attributes) and Table 6.1 presents the item design according to the ten-benchmark Q-matrix.

For the ten attributes, 33 out of 86 items (38%) involved single attribute, A_2, A_3, A_7, A_8, and A_9 (blocks 1 to 5) while the rest of 53 items (62%) measured multiple attributes (blocks 6 to 21). Note that the five attributes, A_1, A_4, A_5, A_6, and $A_{\underline{10}}$ (=tenth Attribute) are not measured directly by the test items but jointly with other attributes (e.g., A_{124}, A_{456}, $A_{236\underline{10}}$). There are a total of 21 blocks in this item design but no block exists for π_1, π_4, π_5, π_6, and $\pi_{\underline{10}}$.

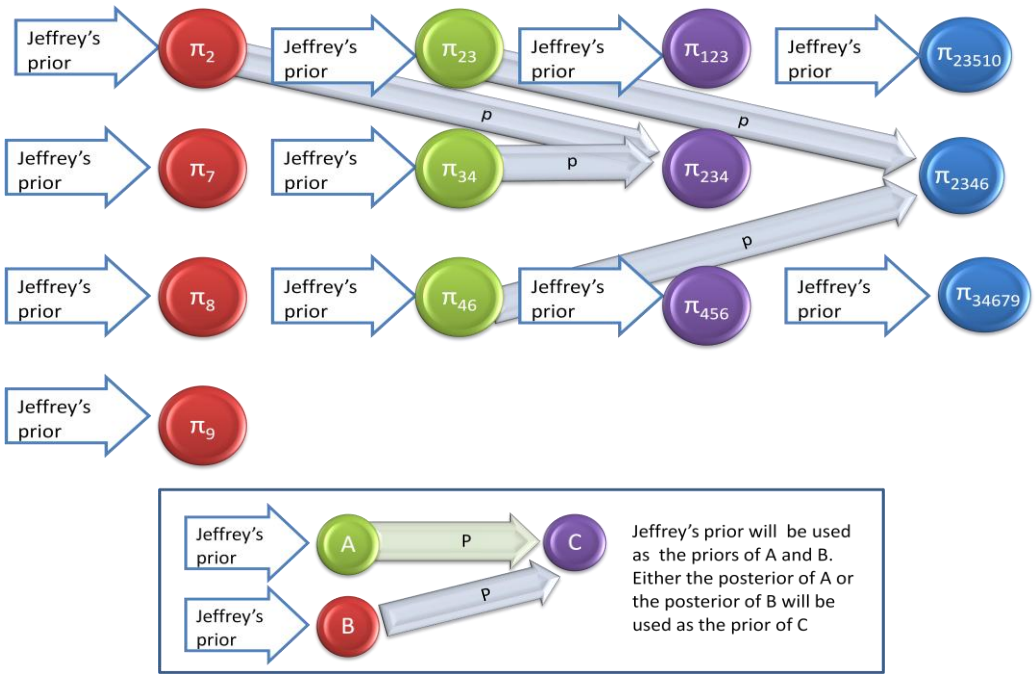
6.1.2 Estimating the Parameters Directly Measured by the Items (Step 1)

Table 6.1 Item Design for the Ten Attributes

	Attribute												
Block	1	2	3	4	5	6	7	8	9	10	*Item #*	*Number of items*	*Parameter*
1	0	1	0	0	0	0	0	0	0	0	71~74	4	π_2
2	0	0	1	0	0	0	0	0	0	0	53	1	π_3
3	0	0	0	0	0	0	1	0	0	0	19~23, 43~46	9	π_7
4	0	0	0	0	0	0	0	1	0	0	75~82	8	π_8
5	0	0	0	0	0	0	0	0	1	0	36~42, 54~57	11	π_9
6	0	1	1	0	0	0	0	0	0	0	6, 12	2	π_{23}
7	0	0	1	1	0	0	0	0	0	0	65~70	6	π_{34}
8	0	0	0	1	1	0	0	0	0	0	24	1	π_{45}
9	0	0	0	1	0	1	0	0	0	0	31, 32, 34, 35, 83~86	8	π_{46}
10	0	0	0	1	0	0	1	0	0	0	47	1	π_{47}
11	0	0	1	0	0	1	0	0	0	0	52	1	π_{36}
12	1	1	1	0	0	0	0	0	0	0	8~11	4	π_{123}
13	0	1	1	1	0	0	0	0	0	0	2~5	4	π_{234}
14	1	1	0	1	0	0	0	0	0	0	1	1	π_{124}
15	0	1	1	0	0	0	0	0	0	1	14	1	$\pi_{23\underline{10}}$
16	0	0	0	1	1	1	0	0	0	0	25~30, 33	7	π_{456}
17	1	1	1	1	0	0	0	0	0	0	7	1	π_{1234}
18	0	1	1	0	0	1	0	0	0	1	13	1	$\pi_{236\underline{10}}$
19	0	1	1	0	1	0	0	0	0	1	15~18	4	$\pi_{235\underline{10}}$
20	0	1	1	1	0	1	0	0	0	0	48~51	4	π_{2346}
21	0	0	1	1	0	1	1	0	1	0	58~64	7	π_{34679}

Note: 1 = measuring, 0 = not measuring

As the first step, the four single-attribute parameters (π_2, π_7, π_8, & π_9) and the nine multiple-attribute parameters (π_{23}, π_{34}, π_{46}, π_{123}, π_{234}, π_{456}, π_{23510}, π_{2346}, & π_{34679}) were estimated (see Table 6.1). They could be estimated directly by the item responses in the same way used in Step 1 and Step 2 in the previous study 1. The procedure of estimating the posteriors of these thirteen parameters is graphically presented in Figure 6.1. It should be noted that the eight blocks (2, 8, 10, 11, 14, 15, 17, & 18) which are shaded in Table 6.1 include only one item each. Thus, the items of these eight blocks were excluded in this step because using only one item for the parameter estimation could be misleading and no prior information of the parameter was available.



Firgure 6.1. Estimating the Posteriors of the thirteen Parameters.

As in Real Data Study 1, Jeffreys' prior, *beta*(.5,.5), was used as the priors of π_2, π_7, π_8, π_9, π_{23}, π_{34}, π_{46}, π_{123}, π_{456}, $\pi_{235\underline{10}}$, and π_{34679} because no prior knowledge about these eleven parameters was available from the data. However, for π_{234}, since the posteriors of π_2 and π_{34} could provide the prior knowledge, the posterior of either π_2 or π_{34} which had the smaller mean was used as the prior of π_{234} in the same way used in Step 2 of Real Data Study 1. Likewise, the posterior of either π_{23} or π_{46} with the smaller mean was used as the prior for the parameter π_{2346} estimation.

6.1.3 Inference about the Unmeasured Single-Attribute Parameters (Step 2)

In the previous step, the four single-attribute parameters (π_2, π_7, π_8, & π_9) were estimated along with the nine multiple-attribute parameters. Six parameters out of the ten single-attribute parameters (π_1 through π_{10}) were still not estimated; of the six, five parameters (π_1, π_4, π_5, π_6, & $\pi_{\underline{10}}$) had no test item that directly measured them and π_3 was involved by only one item. Therefore, the purpose of this step was to find a way to infer the unmeasured single-attribute parameters from the thirteen parameters estimated in Step 1. Some single-attribute parameters could be inferred by a simple and obvious way, but some of the parameters should be inferred by a complex or many alternate ways due to the item design of this mathematical test. For example, π_5 could be inferred by the posteriors of π_{46} and π_{456} as follows: If an examinee showed an evidence of mastery of A_{46} ($\hat{\pi}_{46} \geq .7$) but A_5 was not measured for the examinee in Step 1, then it would be obvious that the mastery (or non-mastery) of A_{456} depends on whether the examinee mastered or not A_5. Thus, it was inferred that the posterior of π_{456} would be very close to the unknown posterior of π_5. In such a case, $\hat{\pi}_{456}$ was used as the approximation of $\hat{\pi}_5$. However, if an examinee did not master A_{46}, then A_{456} should not be mastered. In such

case, $\hat{\pi}_{456}$ would not be used as $\hat{\pi}_5$ because it was uncertain whether the non-mastery of A_{456} was due to the non-mastery of A_{46} only or both A_{46} and the unknown A_5. Since π_5 was still unknown in this case, Jeffreys' prior remained as the posterior of π_5 without any inference about it. It should be noted that such logical processes for the inference can be more complex for the parameters of some attributes depending on the test item design. Also, there would be a variety ways of the inference for some attributes such as A_3, A_4 that are jointly measured with many other attributes (e.g., A_{23}, A_{34}, A_{46}, A_{234}, A_{2346}). Therefore, this step was performed as an effort to find a way to infer the unmeasured parameters in Step 1 (π_1, π_3, π_4, π_5, π_6, & $\pi_{\underline{10}}$). Followings are the inference processes of the six parameters:

- To infer π_1, if $\hat{\pi}_{23} \geq 0.7$ (mastery) or $\hat{\pi}_{234} \geq 0.7$ (mastery), then $\hat{\pi}_{123}$ was used as $\hat{\pi}_1(\hat{\pi}_1 = \hat{\pi}_{123})$. Otherwise $\hat{\pi}_1 = 0.5$ (the mean of Jeffreys' prior).
- To infer π_4, if $\hat{\pi}_{23} \geq 0.7$ or $\hat{\pi}_{123} \geq 0.7$ (mastery), then $\hat{\pi}_4 = \hat{\pi}_{234}$. Else if $\hat{\pi}_{46} \geq 0.7$), then $\hat{\pi}_4 = \hat{\pi}_{46}$. Otherwise $\hat{\pi}_4 = 0.5$.
- To infer π_5, if $\hat{\pi}_{46} \geq 0.7$, then $\hat{\pi}_5 = \hat{\pi}_{456}$. Otherwise $\hat{\pi}_5 = 0.5$.
- To infer π_6, if $\hat{\pi}_{234} \geq 0.7$, then $\hat{\pi}_6 = \hat{\pi}_{2346}$. Else if all of $\hat{\pi}_{34}$, $\hat{\pi}_7$, and $\hat{\pi}_9 \geq 0.7$, then $\hat{\pi}_6 = \hat{\pi}_{34679}$. Otherwise $\hat{\pi}_6 = 0.5$.
- To infer $\pi_{\underline{10}}$, if either $\hat{\pi}_{123}$ or $\hat{\pi}_{234} \geq 0.7$ and $\hat{\pi}_{456} \geq 0.7$, then $\hat{\pi}_{\underline{10}} = \hat{\pi}_{235\underline{10}}$. Otherwise $\hat{\pi}_6 = 0.5$.
- To update the posterior of π_3, if $\hat{\pi}_2 \geq 0.7$, then use the response data of A_{23}. Else if $\hat{\pi}_{46} \geq 0.7$ or $\hat{\pi}_4 \geq 0.7$, then use the response data of A_{34}.

For π_1, if the examinee mastered A_{23} ($\hat{\pi}_{23} \geq 0.7$) or A_{234} ($\hat{\pi}_{234} \geq 0.7$), then $\hat{\pi}_{123}$ was used as the π_1 estimate ($=\hat{\pi}_1$). Otherwise, the mean of Jeffreys' prior was used as $\hat{\pi}_1 (= 0.5)$.

For the rest of parameters except π_3, similar rules were applied as shown above. The posterior of π_3 could be estimated by the Jeffreys' prior and its item response data as in Step 1. However, since a single item involved A_3, the posterior of π_3 should be updated by the response data to the relevant multiple-attribute items as in Step 3 of Study 1. Therefore, item response data of A_{34} was used for updating the posterior of π_3 if an examinee was diagnosed to master A_{46} or A_4.

6.2 Results

6.2.1 Descriptive Statistics

The descriptive statistics of the raw score (same as in Real Data Study 1) and the thirteen $\hat{\pi}_k$'s estimated in Step 1 and the six $\hat{\pi}_k$'s obtained in Step 2 were provided in Table 6.2. Note that the six parameters ($\hat{\pi}_1$, $\hat{\pi}_3$, $\hat{\pi}_4$, $\hat{\pi}_5$, $\hat{\pi}_6$, & $\hat{\pi}_{\underline{10}}$) could not be directly measured by the test items, thus they were inferred from the parameter estimates in Step 1. In Real Data Study 1, Standard 3 (Geometry) showed highest mean $\hat{\pi}_k$ while Standard 1 (Number and Computation) and Standard 4 (Data) had lowest $\hat{\pi}_k$ of the four single attributes. In this study, A_7 (Geometric Figures and their Properties) and A_8 (Geometry from an Algebraic Perspective) which are the benchmarks of Standard 3 (see Appendix A) also showed the highest mean $\hat{\pi}_k$'s (.69 and .73, respectively) of the ten single attributes. On the other hand, A_1 (Number Sense), A_2 (Number System and their Properties), and A_3 (Computation) which are the benchmarks of Standard 1 showed low mean $\hat{\pi}_k$'s; .48. .66, and .63, respectively.

Table 6.2 *Descriptive Statistics of the Raw Score and $\hat{\pi}_k$ from Step 1 and Step 2 (N=2993)*

		Min.	*Max.*	*M*	*SD*
Step 1	Raw score	22	86	60.68	15.12
	$\hat{\pi}_2$	0.1000	0.9000	0.6572	0.2431
	$\hat{\pi}_7$	0.0500	0.9500	0.6948	0.2449
	$\hat{\pi}_8$	0.0556	0.9444	0.7319	0.2043
	$\hat{\pi}_9$	0.0417	0.9583	0.6394	0.1968
	$\hat{\pi}_{23}$	0.1667	0.8333	0.5043	0.2480
	$\hat{\pi}_{34}$	0.0714	0.9286	0.6796	0.2139
	$\hat{\pi}_{46}$	0.1000	0.9000	0.7662	0.1889
	$\hat{\pi}_{123}$	0.1000	0.9000	0.6706	0.2110
	$\hat{\pi}_{234}$	0.0455	0.9444	0.6322	0.1840
	$\hat{\pi}_{456}$	0.0625	0.9375	0.6142	0.2456
	$\hat{\pi}_{235\underline{10}}$	0.1000	0.9000	0.7267	0.2137
	$\hat{\pi}_{2346}$	0.0556	0.9286	0.5802	0.2083
	$\hat{\pi}_{34679}$	0.0625	0.9375	0.7451	0.1675
Step 2	$\hat{\pi}_1$	0.1000	0.9000	0.6301	0.1771
	$\hat{\pi}_3$	0.2500	0.7500	0.4795	0.1083
	$\hat{\pi}_4$	0.0556	0.9444	0.6606	0.1750
	$\hat{\pi}_5$	0.0625	0.9375	0.6313	0.2172
	$\hat{\pi}_6$	0.0714	0.9375	0.5964	0.1583
	$\hat{\pi}_{\underline{10}}$	0.1000	0.9000	0.6162	0.1739

p_k was also presented for each attribute in Table 6.3. A_8 showed the highest p_k (=.66) and A_6 had the lowest value (.24) of the single attributes. Unexpectedly, some of the multiple attributes (e.g., A_{46}, $A_{235\underline{10}}$) showed higher values than single attributes. It may suggest that the single-attribute parameters, especially the inferred attributes were underestimated due to the lack of information for the inference.

Table 6.3 *Attribute Difficulty*

Attribute	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	$A_{\underline{10}}$
p_k	.41	.63	.43	.48	.36	.24	.56	.66	.44	.33
Attribute	A_{23}	A_{34}	A_{46}	A_{123}	A_{234}	A_{456}	$A_{235(10)}$	A_{2346}	A_{34679}	
p_k	.28	.51	.83	.65	.40	.38	.78	.29	.58	

Inter-attribute correlations were also provided in Table 6.4 (the ten single-attributes) and in Table 6.5 (between the single and the multiple attributes).

Table 6.4 *Inter-Attribute Correlations of the Single Attributes*

	Standard 1			*Standard 2*			*Standard 3*		*Standard 4*	
	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_5$	$\hat{\pi}_6$	$\hat{\pi}_7$	$\hat{\pi}_8$	$\hat{\pi}_9$	$\hat{\pi}_{\underline{10}}$
$\hat{\pi}_1$	1	.470	.286	.521	.490	.600	.509	.466	.528	.530
$\hat{\pi}_2$		1	.317	.607	.360	.372	.441	.434	.425	.362
$\hat{\pi}_3$			1	.338	.280	.267	.418	.385	.361	.266
$\hat{\pi}_4$				1	.375	.460	.460	.451	.473	.391
$\hat{\pi}_5$					1	.502	.542	.487	.500	.705
$\hat{\pi}_6$						1	.526	.449	.574	.520
$\hat{\pi}_7$							1	.609	.602	.475
$\hat{\pi}_8$								1	.535	.441
$\hat{\pi}_9$									1	.480

* All correlations are significant at the 0.001 level.

Table 6.5 *Inter-Attribute Correlations between Single Attributes and Multiple Attributes*

	$\hat{\pi}_{23}$	$\hat{\pi}_{34}$	$\hat{\pi}_{46}$	$\hat{\pi}_{123}$	$\hat{\pi}_{234}$	$\hat{\pi}_{456}$	$\hat{\pi}_{23\underline{10}}$	$\hat{\pi}_{2346}$	$\hat{\pi}_{34679}$
$\hat{\pi}_1$	.539	.520	.358	.698	.606	.509	.398	.612	.430
$\hat{\pi}_2$	.309	.401	.320	.355	.720	.404	.349	.442	.381
$\hat{\pi}_3$	.243	.361	.430	.324	.367	.380	.342	.369	.336
$\hat{\pi}_4$	.189	.598	.471	.301	.840	.442	.375	.414	.429
$\hat{\pi}_5$	.382	.455	.365	.453	.473	.927	.393	.549	.454
$\hat{\pi}_6$	.556	.536	.346	.451	.525	.496	.342	.747	.438
$\hat{\pi}_7$	.413	.556	.486	.525	.571	.596	.497	.626	.532
$\hat{\pi}_8$	.370	.531	.464	.484	.549	.545	.480	.549	.523
$\hat{\pi}_9$	.428	.535	.455	.506	.566	.548	.453	.615	.496
$\hat{\pi}_{\underline{10}}$	.383	.438	.321	.465	.477	.700	.468	.525	.402

* All correlations are significant at the 0.001 level.

Overall, the inter-attribute correlations (M = .469) were lower than those of the four-attribute data in Real Data Study 1 (M = .792). It was speculated that the ten benchmarks are more specific (unique) contents than the four standards, thus they were less inter-

correlated. The intercorrelations of the ten single attributes (A_1 through A_{10}) showed low to medium correlations ranging from .280 to .705 (M = .460).

The correlations between the single attributes and the nine multiple-attributes (A_{23} through A_{34679}) showed a wide range of correlations from .189 to .927 (M = .473). In Table 6.4, it was observed that the inter-benchmark correlations within standards were not higher than those between standards. For example, the three benchmarks (A_1, A_2, A_3) of Standard 1 showed lower intercorrelations (r = .286 to .470, M = .358) than the correlations with A_7 (r = .508, .441, and .418, respectively) that is one of the benchmarks of Standard 3. Therefore, it was speculated that the specific benchmarks of each standard were unique, thus generally they were less related with each other than the benchmarks outside of the standard. It may suggest that the higher inter-benchmark correlations between standards reflected the relatively high inter-standard correlations as shown in Real Data Study 1 (see Table 6.4 & 6.5).

6.2.2 Individual diagnosis result

The BIBP method estimated $\hat{\pi}_k$ and α_k of a total of nineteen attributes (A_1, A_2, A_3, A_4,…, A_{34679}) for each examinee. Table 6.6 shows the diagnosis results for the same three examinees with the raw score of 64 who were presented in Real Data Study 1 (see Table 5.6); $\hat{\pi}_k$ and α_k of the ten single attributes (ten benchmarks).

Examinee #116 who was diagnosed to have mastered only Standards 2 and 3 (α_k = 0110) in Study 1 showed the mastery pattern of the ten benchmarks of α_k = 0000101101 (=mastery of Benchmarks 5, 7, 8, & 10). Consistently with the diagnosis result of the standards in Study 1, this examinee showed no mastery of A_1, A_2, and A_3 (=

the three benchmarks of Standard1) and A_{10} (= one of the benchmarks of Standard 4) but mastery of both A_7 and A_8 (the two benchmarks of Standard 3).

Table 6.6 *Diagnosis Results for the Three Examinees (Raw Score: 64/86)*

		Standard 1			*Standard 2*			*Standard 3*		*Standard 4*	
	Examinee	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_5$	$\hat{\pi}_6$	$\hat{\pi}_7$	$\hat{\pi}_8$	$\hat{\pi}_9$	$\hat{\pi}_{10}$
$\hat{\pi}_k$	#116	0.50	0.30	0.56	0.61	0.94	0.50	0.75	0.83	0.38	0.90
	sd	0.35	0.19	0.17	0.15	0.08	0.35	0.13	0.12	0.13	0.12
	#130	0.50	0.10	0.69	0.90	0.69	0.69	0.75	0.83	0.96	0.50
	sd	0.35	0.12	0.15	0.12	0.15	0.15	0.13	0.12	0.06	0.35
	#2706	0.90	0.90	0.94	0.83	0.50	0.72	0.95	0.72	0.54	0.50
	sd	0.12	0.12	0.08	0.12	0.35	0.14	0.07	0.14	0.14	0.35
α_k	#116	0	0	0	0	1	0	1	1	0	1
	#130	0	0	0	1	0	0	1	1	1	0
	#2706	1	1	1	1	0	1	1	1	0	0

sd = posterior standard deviation

However, for Standard 2 that this examinee was diagnosed to have mastered showed mastery of only A_5 (one of the three benchmarks of Standard 2) as shown in Table 6.6 ($\hat{\pi}_5$ = .94). For the other two benchmarks (A_4 and A_6) of Standard 2, $\hat{\pi}_k$ was lower than .70 with relatively large posterior *sd* (.15 and .35, respectively). It should be noted that these three parameters (π_4, π_5, & π_6) were inferred using the posteriors of the parameters estimated in Step 1 since there was no test item to directly measure them. The inconsistency between the diagnosis results regarding Standard 2 and Benchmarks 4 and 6 may be due to the lack of information for the inference in Step 2. The estimated posteriors of the three benchmarks of Standard 2 for examinee #116 were shown in Figure 6.2.

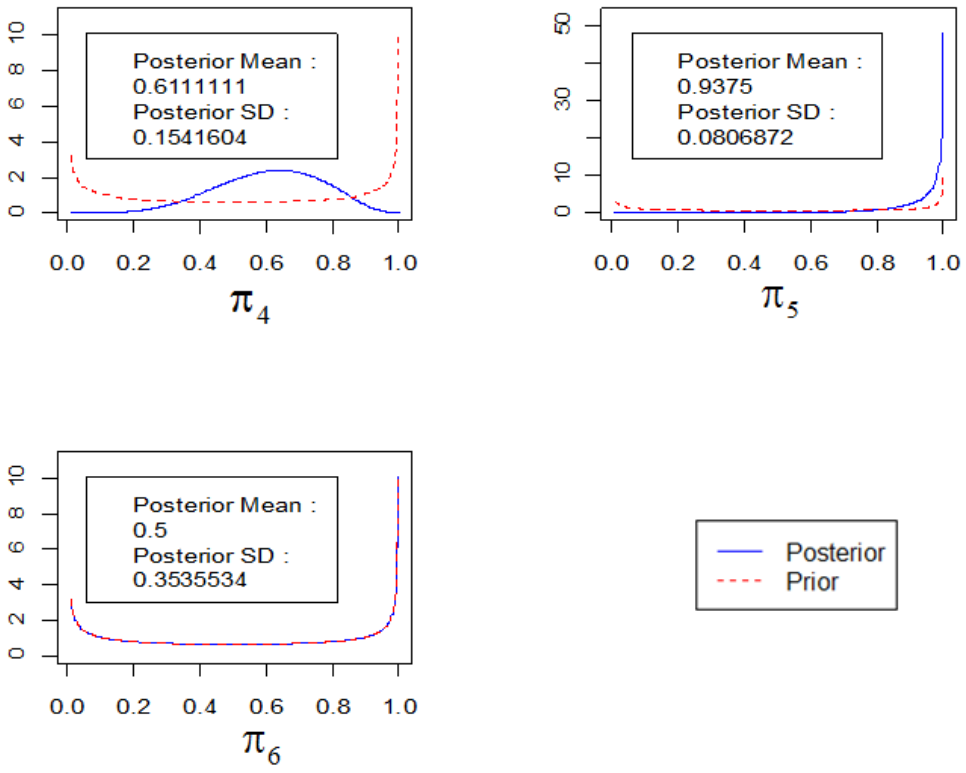


Figure 6.2 Estimated Posteriors of π_4, π_5, and π_6 for Examinee #116

Examinee #130 who was diagnosed to have mastered Standards 2, 3, and 4 (α_k = 0111) in Real Data Study 1 showed the mastery pattern of the ten benchmarks as α_k = 0001001110 (= mastery of Benchmarks 4, 7, 8, & 9). The diagnosis result regarding the seven attributes (A_1, A_2, A_3, A_4, A_7, A_8, and A_9) was consistent with the diagnosis result of the four standards in Real Data Study 1. Although Benchmarks 5, 6, and 10 should be mastered based on the Study 1 result, this examinee nearly mastered Benchamrks 5 and 6 (π_k =.69) and did not show the mastery of Benchmark 10 due to the lack of information for the inference (π_k =.50). The estimated posteriors of π_4, π_6, and π_{10} for the for examinee #130 were shown in Figure 6.3.

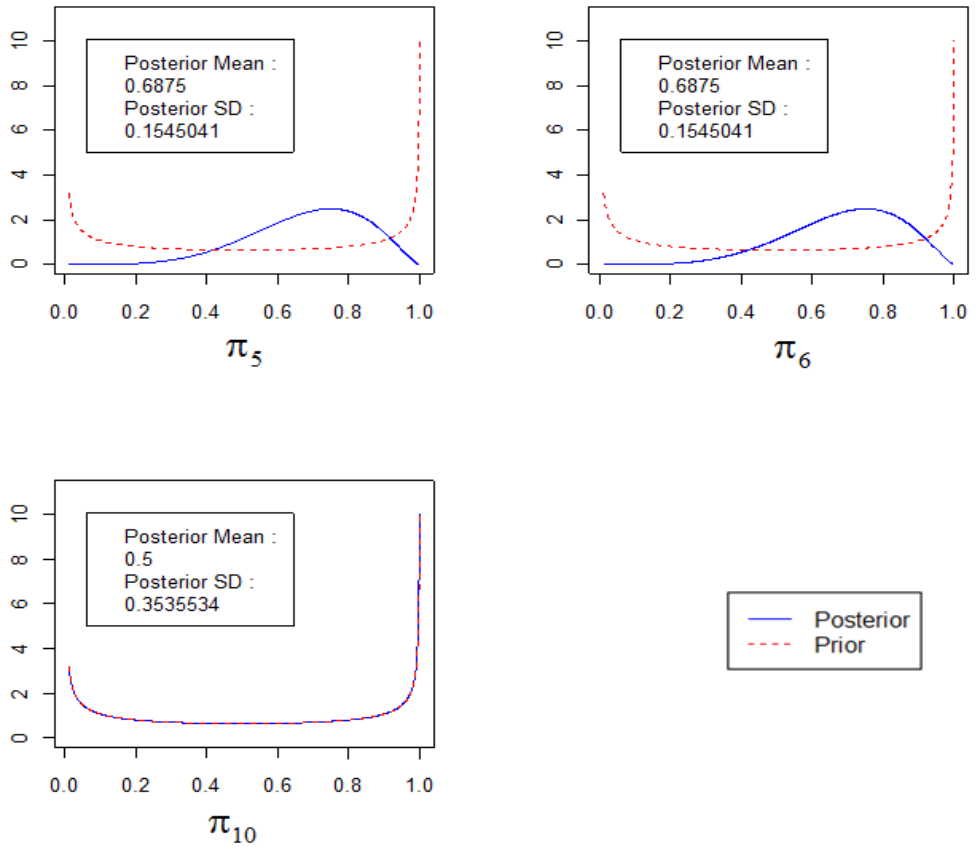


Figure 6.3 Estimated Posteriors of π_4, π_5, and π_9 for Examinee #130

In Real Data Study 1, examinee #2706 showed the four-standard mastery pattern of α_k = 1010 in Step 1&2 and α_k = 1110 in Step 3. In this study, this examinee showed the mastery of all the benchmarks (A_1 through A_8) of Standards 1, 2, and 3, except A_1, A_5; no inference was made about π_5 for this examinee due to the lack of information ($\hat{\pi}_5$ = .50, *sd* = .35). The estimated posteriors of Standards 1 and 2 are presented in Figure 6.4.

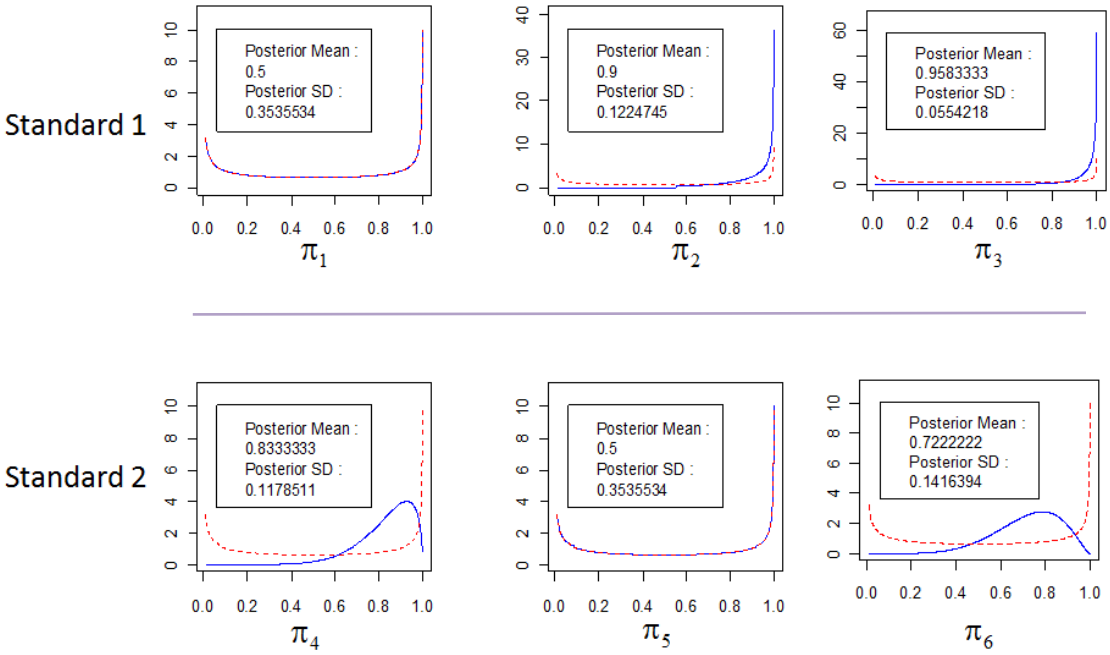


Figure 6.4 Estimated Posteriors of π_1, π_2, π_3, π_4, π_5, and π_6 for Examinee #2706

6.3 Discussion

In this study, the BIBP method was applied to a more complex test data which involved ten attributes. In this item design, every attribute was not directly measured by the test items. Therefore, it was explored how to infer π_k of such unmeasured single attributes from the parameter estimates of the attributes directly measured by the test items. There will be different ways of the inference because some attributes were jointly measured with many other attributes (e.g., A_{234}, A_{2346}, A_{456}, A_{23510}, and A_{34679}). Logical processes for the inference were introduced and applied to six unmeasured attributes (A_1, A_3, A_4, A_5, A_6, and A_{10}). It seemed that some of the attributes were underestimated due to the lack of information for the inference. The inference processes need to be revised in the future study.

CHAPTER 7

REAL DATA STUDY 3: COMPARING BIBP TO DINA AND LCDM

In Real Data Study 1 and 2, the BIBP method was applied to a mathematical test data to estimate examinees' attribute-mastery probability (π_k) and the pattern (α_k) of each examinee. In this study, the estimated parameters by the BIBP method were compared to the parameters estimated by existing diagnosis models, DINA and LCDM. As elaborated in Chapter 2, DINA is a noncompensatory model which is appropriate to diagnose mathematical skills where mastery of the skills cannot compensate for nonmastery of the other skills. In DINA, examinees are classified into two classes for each item; those who have mastered all the skills required to solve the item (ξ_{ij} =1 in Equation 2.3) and those who do not (ξ_{ij} =0). Although it is one of the most widely accepted noncompensatory models, one criticism of DINA is that this model may not be very practical because missing one of the required attributes is considered to be equivalent to missing all the attributes (Henson & Douglas, 2005). LCDM is a generalized model which can be applied to any type of data, whether compensatory or non-compensatory. It provides information about whether conditional relationships exist between required attributes and the item response, thus it can give the insight as to what type of model (compensatory or non-compensatory) could be more appropriate for the items (Henson et al., 2009).

7.1 Method

7.1.1 Subjects and Instruments

The same subjects and instruments as in Real Data Study 1 and Study 2 were used for the comparison purpose in this study. The item design involving four attributes (see Table 5.1) was adopted for this study since the estimated item parameters of LCDM was expected to have large standard errors especially for large number (e.g., ten) of attributes involved in the test items (Henson et al., 2009).

7.1.2 Procedure

For the parameter estimation of the both models, the LCDM program (Burke & Henson, 2008) was used. The program allows user specified constraints for LCDM to fit the DINA model. For the DINA model, only the highest order interaction parameter and the intercept will be included under the LCDM framework (see Equation 2.17) since it is the non-compensational model, thus no main effect needs to be estimated. For example, the logit, $\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$, of the LCDM and DINA for item #1 which involves A_1 and A_2 are as follows:

LCDM: $\boldsymbol{\lambda}_1^T \boldsymbol{h}(\boldsymbol{q}_1, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,1,(1)}(\alpha_{r1}) + \lambda_{i,1,(2)}(\alpha_{r2}) + \lambda_{i,2,(1,2)}(\alpha_{r1}\alpha_{r2})$

DINA: $\boldsymbol{\lambda}_1^T \boldsymbol{h}(\boldsymbol{q}_1, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,2,(1,2)}(\alpha_1\alpha_2)$

Followings are the logits of both models for item #15 involving A_1, A_2, and A_4:

LCDM: $\boldsymbol{\lambda}_{15}^T \boldsymbol{h}(\boldsymbol{q}_{15}, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,1,(1)}(\alpha_1) + \lambda_{i,1,(2)}(\alpha_2) + \lambda_{i,1,(4)}(\alpha_4)$

$+\lambda_{i,2,(1,2)}(\alpha_1\alpha_2) + \lambda_{i,2,(1,4)}(\alpha_1\alpha_4) + \lambda_{i,2,(2,4)}(\alpha_2\alpha_4) + \lambda_{i,3,(1,2,4)}(\alpha_1\alpha_2\alpha_4),$

DINA: $\boldsymbol{\lambda}_{15}^T \boldsymbol{h}(\boldsymbol{q}_{15}, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,3,(1,2,4)}(\alpha_1\alpha_2\alpha_4)$

For this item, DINA includes only three-way interaction of A_1, A_2, and A_4 and the intercept without any main effect nor two-way interactions. Therefore, DINA has two

parameters per item regardless of the number of attributes involved in the item. However, in LCDM, the number of parameters per item became doubled as one attribute is added (e.g., two parameters for one attribute, four parameters for two attributes, eight for three attributes, and sixteen for four attributes).

After estimating attribute difficulty attribute mastery probability (π_k) and the attribute mastery pattern (α_k) of each examinee, the estimation results of the three models, DINA, LCDM, and BIBP, were compared. Also, the proportion of examinees who have mastered attribute *k* (= p_k) and inter-attribute correlations were obtained based on the parameter estimates of the three models and compared.

7.2 Results

7.2.1 Descriptive Statistics

The descriptive statistics of the raw score and $\hat{\pi}_k$ of the four attributes were provided in Table 7.1. Each descriptive statistic was reported separately for the DINA, LCDM, and BIBP estimations for the comparison purpose. However, the raw score statistics are same for the three models. As shown in Table 7.1, the estimated attribute-mastery probability of the four attributes (A_1, A_2, A_3, A_4) were slightly different among the three models; $\hat{\pi}_1$ = .56, $\hat{\pi}_2$ = .62, $\hat{\pi}_3$ =.84, and $\hat{\pi}_4$ = .95 for DINA, ; $\hat{\pi}_1$ = .48, $\hat{\pi}_2$ = .59, $\hat{\pi}_3$ =.54, and $\hat{\pi}_4$ = .51for LCDM, and ; $\hat{\pi}_1$ = .64, $\hat{\pi}_2$ = .67, $\hat{\pi}_3$ =.72, and $\hat{\pi}_4$ = .64 for BIBP (Step 1&2 result). The average $\hat{\pi}_k$-value was the highest (.746) for DINA and was the lowest (.531) for CDM while it was in the middle (.668) for BIBP of the three estimation results.

Table 7.1 *Descriptive Statistics of the Raw Score and $\hat{\pi}_k$ of the Four Attributes for DINA, LCDM, and BIBP (N=2993)*

	DINA			LCDM			BIBP		
	Min	*Max*	*M*	*Min*	*Max*	*M*	*Min*	*Max*	*M*
Raw score	22	86	60.68 (15.12)	22	86	60.68 (15.12)	22	86	60.68 (15.12)
$\hat{\pi}_1$	0.00	1.00	0.564 (0.47)	0.00	1.00	0.481 (0.47)	0.042	0.958	0.639 (0.21)
$\hat{\pi}_2$	0.00	1.00	0.623 (0.46)	0.00	1.00	0.588 (0.46)	0.088	0.971	0.670 (0.20)
$\hat{\pi}_3$	0.00	1.00	0.843 (0.36)	0.00	1.00	0.539 (0.48)	0.083	0.972	0.724 (0.21)
$\hat{\pi}_4$	0.00	1.00	0.952 (0.21)	0.00	1.00	0.514 (0.45)	0.042	0.958	0.639 (0.20)

Note. ( ) =*standard deviation*

Note that the standard deviation of $\hat{\pi}_k$ was smallest in the BIBP estimation. In the DINA estimation, $\hat{\pi}_3$ (=.84) and $\hat{\pi}_4$ (=.95) appeared to be overestimated, considering that average raw scores for the items involving A_3 and A_4 were 12.5/17 (74%) and 7.2/11 (65%), respectively. Inter-attribute correlations and the proportion of examinees who have mastered attribute *k* (p_k) were presented in Table 7.2 for the three models. It should be noted that the calculation of inter-attribute correlations in DINA and LCDM is different from that in BIBP. In the DINA and LCDM calculation, each attribute was assumed to load on a general factor and the correlation between any two attributes was the product of the loadings of the two respective attributes on the general factor (Burke & Henson, 2008). However, in BIBP, the Pearson correlation between any two $\hat{\pi}_k$was used as the inter-attribute correlation. The inter-attribute correlations of DINA were high, ranging from .751 to .996 (*M* = .869) while the inter-attribute correlations of LCDM and BIBP had medium sizes, ranging from .617 to .755 (*M* = .685) for LCDM and from .614 to .722 (*M* = .661) for BIBP.

Table 7.2 *Inter-Attribute Correlations and* p_k

Model	Attribute	A_1	A_2	A_3	A_4
DINA	A_1	1	0.751	0.8	0.798
	A_2		1	0.936	0.934
	A_3			1	0.996
	p_k	0.64	0.72	0.95	0.99
LCDM	A_1	1	0.617	0.745	0.675
	A_2		1	0.69	0.626
	A_3			1	0.755
	p_k	0.52	0.60	0.57	0.55
BIBP	A_1	1	0.672	0.687	0.614
	A_2		1	0.722	0.632
	A_3			1	0.637
	p_k	0.47	0.49	0.57	0.44

The p_k results were consistent with the $\hat{\pi}_k$ estimation results in Table 7.1. In DINA, A_3 (Geometry) and A_4 (Data) showed very high proportions of the attribute mastery, p_3 = .95 and p_4 = .99, respectively ($\hat{\pi}_3$= .84, and $\hat{\pi}_4$= .99). For both LCDM and BIBP, A_2 (Algebra) and A_3 (Geometry) showed higher p_k's than A_1 (Number and Computation) and A_4 (Data). Also, p_3 was same (=.57) between the two models. As discussed in Chapter 5, in DINA and LCDM, 0.5 was used as the cutoff $\hat{\pi}_k$-value for deciding the mastery or nonmastery of attribute *k* while 0.7 was used in the BIBP method.

7.2.2 Individual Diagnosis Result

Table 7.3 presents the comparison of the three models in estimating the examinee attribute mastery pattern (α_k) for each attribute and for whole pattern. For α_1 and α_2, the proportion of same classifications was high among the three models, ranging .86 to .94. However, for α_3 and α_4, same classification proportion was high only between LCDM and BIBP (.95 and .87, respectively). The proportion of whole pattern was .67 between LCDM and BIBP.

Table 7.3 *Proportion of Same Classifications for Examinee Attribute Mastery Pattern (α_k) of the Three Models (N=2993)*

Comparison	α_1	α_2	α_3	α_4	Whole pattern (α_1 α_2 α_3 α_4)
DINA vs. LCDM	0.90	0.94	0.70	0.56	0.47
DINA vs. BIBP	0.87	0.86	0.72	0.49	0.34
LCDM vs. BIBP	0.90	0.89	0.95	0.87	0.67

For the three examinees (raw score: 64 out of 86) who were examined as sample subjects in Study 1 and 2, their attribute mastery probabilities and the mastery patterns were compared among the DINA, LCDM, and BIBP estimations in Table 7.4.

Table 7.4 *Diagnosis Results ($\hat{\pi}_k$, α_k) for the Three Examinees (Raw Score: 64/86) by the Three Models*

Model	*Examinee ID*	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	α_k (A_1	A_2	A_3	A_4)
DINA	#116	0.920	1	1	1	1	1	1	1
	#130	0	1	1	1	0	1	1	1
	#2706	1	1	1	1	1	1	1	1
LCDM	#116	0	1	0.727	0	0	1	1	0
	#130	0	1	1	1	0	1	1	1
	#2706	1	0.290	1	0.263	1	0	1	0
BIBP	#116	0.458	0.912	0.806	0.375	0	1	1	0
	#130	0.208	0.794	0.806	0.958	0	1	1	1
	#2706	0.875	0.500	0.861	0.542	1	0	1	0

Examinee #116 was diagnosed to have mastered all the four standards by the DINA model but diagnosed to have mastered only two standards (A_2, A_3) by the LCDM and BIBP method. However, the α_k -diagnosis result of examinee #130 for the three models were same: mastery of three standards (A_2, A_3, A_4) but no mastery of A_1.

Examinee #2706 was diagnosed to have mastered all the standards by DINA but to have mastered A_1 and A_3 only by LCDM and BIBP. Overall, the α_k -diagnosis results were same between the LCDM and BIBP estimation although $\hat{\pi}_k$-values (in Table 5.10) were not identical. However, the diagnosis result of DINA was quite different from those of the other two models except examinee #130.

7.3 Discussion

In this study, it was found that there were differences in the attribute mastery probability estimates ($\hat{\pi}_k$) among the three model (DINA, LCDM, BIBP), which could result in different attribute mastery pattern (α_k)- diagnosis results. However, it remains uncertain which model provides more accurate diagnosis information about examinees before conducting a simulation study using true examinee parameters known.

As expected, one big advantage of the BIBP method over DINA and LCDM was easy and quick parameter estimation. For this four-attribute data, it took less than a second to estimate $\hat{\pi}_k$ and α_k for all the 2993 examinees by the BIBP method while it took about 30 minutes by DINA and about one and a half hours by LCDM using the Pentium(R) dual-core CPU (1.50 GHz). It was because the BIBP estimation needs no iteration procedure such as MMLE-EM or MCMC. However, the LCDM program used the MCMC algorithm for the DINA and LCDM parameter estimations. Especially for LCDM, although the number of parameters per item is 16 for the current four-attribute data, it will become doubled as one attribute is added. Therefore, for ten-attribute data, it will become 16×2^6 (= 1024) parameters per item, which will greatly increase the computational demand and time.

Another advantage of the BIBP method is to provide the estimates of both single-attribute parameters (e.g., π_1, π_2, π_3) and multiple- (or combined) attribute parameters (e.g., π_{12}, π_{23}, π_{123}) based on the assumption that the combined attributes are not necessarily same as simply adding the single attributes (see Chpater 4). The separate parameter estimations would be helpful in diagnosing examinees' ability, especially when the attributes within a test items are compensatory. As discussed in Chapter 5, aberrant item response patterns on between single-attributes and the multiple-attributes can be often found in practice (e.g., poor performance on A_1 but better performance on A_{12}). Such an aberrant response pattern can be explained not only by positive or negative slip possibility but also by the compensation of A_1 by A_2 within the test item involving A_{12}. Therefore, providing parameter estimates of both single-and multiple attributes would be helpful for diagnosing test items as well as examinees' ability.

CHAPTER 8

SIMULATION STUDY 1: GENERAL ACCURACY AND EFFECTIVENESS OF THE PARAMETER ESTIMATION

Accurate diagnosis is the final goal of every test. To evaluate the general accuracy of BIBP for CDA, Monte Carlo simulation methods was used to generate item response data assuming that the true attribute mastery pattern (α_k) and attribute mastery probability (π_k) for each examinee were known. Following measures were used to check the accuracy of the diagnostic classification and the parameter estimation ($\hat{\pi}_k$): Correct Classification Rate (CCR), Average Signed Biase (ASB), and Root Mean Square Error (RMSE). CCR was computed for examinees' attribute mastery patterns as well as for each attribute marginally (Henson & Douglas, 2005; Templin, Henson, Templin, & Roussos, 2008). ASB and RMSE were used to measure the discrepancy between the true and estimated π_k. Additionally, the parameter estimation was obtained in two ways: with and without updating the posteriors of the single-attribute parameters by the multiple-attribute data (Step 3 of Real Data Study 1). In order to find out whether the updating step actually improved the accuracy of the parameter estimation, CCRs, ASB's, and RMSEs were compared for both estimation results. Excel programming including Visual Basic (VBA) macro was used for this simulation.

8.1 Method

8.1.1 Item Design

To simulate realistic patterns of attributes involved in a test, the four-attribute item design of the real data study 1(Table 5.1) was used. However, the number of items

in each block was balanced unlike the real data in which some attributes were measured by a single item. Table 8.1 presents the number of items and the parameter to be simulated for each of the nine blocks. Blocks six, seven, and eight include six items per each and the rest of the blocks have seven items per each, thus a total of 60 items were generated in this study.

Table 8.1 *Simulated Item Design*

	Attribute				Number	
Block	1	2	3	4	of items	Parameter
1	1	0	0	0	7	π_1
2	0	1	0	0	7	π_2
3	0	0	1	0	7	π_3
4	0	0	0	1	7	π_4
5	1	1	0	0	7	π_{12}
6	0	1	1	0	6	π_{23}
7	1	0	0	1	6	π_{14}
8	1	1	0	1	6	π_{124}
9	1	1	1	1	7	π_{1234}
Total					60	

1=measuring, 0 = not measuring

8.1.2 Examinee Attribute Mastery Probability and Mastery Pattern

10,000 hypothetical examinees who had different attribute mastery probabilities (π_{jk}) and attribute mastery patterns (a_{jk}) for the four attributes were generated. When generating π_{jk}, two criteria are generally considered to be important for an appropriate simulation (Henson & Douglas, 2005): (1) Attribute difficulty (p_k) and (2) Attribute correlation. First, the actual p_k –values of the four attributes in Real Data Study 1 were used: p_1 = .47, p_2 = .49, p_3= .57, and p_4 = .44, respectively (from step 1&2). Theses actual four attributes seemed to have medium difficulties. Second, attributes have typically non-

zero correlations with each other in practice. Attribute correlations vary depending on the degree of multidimensionality of the test. Low attribute correlations indicate a high degree of multidimensionality of the test and vice versa (Templin et al., 2008). In this study, to imitate the actual relations of the four attributes, the correlation matrix $\boldsymbol{R}$ [4×4] of the four attributes of Real Data Study 1 (from step 1&2) were used:

$$\boldsymbol{R}=\begin{bmatrix} 1 & .67 & .70 & .61 \\ .67 & 1 & .72 & .63 \\ .70 & .72 & 1 & .64 \\ .61 & .63 & .64 & 1 \end{bmatrix}.$$

As the first step of this simulation, a random score z_{jk} (for examinee *j* and attribute *k*) was drawn from a multivariate normal (*MVN*) distribution with a mean vector of zeros (***O***) and the correlation matrix $\boldsymbol{R}$ shown above. As the second step, π_{jk} was generated using the z_{jk} score as follows:

$$\begin{cases} \pi_{jk} \sim U(.70, .99), & if\ z_{jk} \leq INVNORM(p_k) \\ \pi_{jk} \sim U(.01, .69), & otherwise \end{cases}$$

where p_1 = .47, p_2 = .49, p_3= .57, and p_4 = .44.

Notice that *INVNORM* (p_k) is a z-score value corresponding to the cumulative probability of p_k under standard normal distribution. For example, *INVNORM* (p_1 = .47) is -0.075, thus $P(z \leq -0.075)$ is .47. If z_{jk} was smaller than or equal to *INVNORM* (p_k), examinee *j* was considered to have mastered attribute *k*. Then, π_k of the examinee was randomly drawn from the uniform (*U*) distribution (.70, .99). Otherwise, π_k was randomly drawn from a *U* (.01, .69) distribution, considering that the examinee have not mastered the attribute. The four attribute mastery probabilities (π_1, π_2, π_3, & π_4) of each examinee was generated using the different p_k-values for the four attributes as described above. It

should be noted that π_{jk} could be neither 0 nor 1 due to the non-zero positive or (i.e., correct guessing) negative slip possibility. As a result, attribute mastery probability matrix [$j \times k$] was generated, where j = 10,000 (examinees) and k = 4 (attributes). Also, attribute mastery pattern matrix [$j \times k$] was generated based on π_{jk} (mastery if $\pi_{jk} \geq 0.7$; non-mastery otherwise).

8.1.3 Item Response Data Generation

Given π_k generated above, binary response (x_{ij}) to item i was generated for examinee j using a *Uniform* distribution, U (0,1) as follows:

$$x_{ij} = \begin{cases} 1\ (= correct) & if\ U(0,1) \leq \pi^*_{jk}, \\ 0\ (= incorrect) & otherwise \end{cases}$$

where $\pi^*_{jk} = \pi_{jk}$ for single attribute-items (block 1 to 4 in Table 8.1), or

the smallest of all involved π_{jk}s for multiple-attributes items (block 5 to 15).

For example, if examinee j had π_{j1} = .90, π_{j2} = .80, π_{j3} = .30, and π_{j4} = .70, then π^*_{jk} was equal to π_{j1} for the items involving only A_1 (block 1) while π^*_{jk} was equal to π_{j3} for the items measuring all four attributes (block 15) because π_{j3} (=.30) was the smallest of the four π_{jk}s involved for this item. As a result, item response matrix [$j \times i$] was generated, where j = 10,000 (examinees) and i = 60 (items).

8.1.4 Estimation of the Examinee Attribute Mastery Pattern and π_k

Given item response data generated above, examinee attribute mastery pattern and the mastery probability for each attribute were estimated using the same way used in Real Data Study 1 (step 1, 2, and 3). CCR and RMSE were used to check the general accuracy of the diagnostic classification and the parameter estimation ($\hat{\pi}_k$), respectively. Additionally, it was examined whether the step 3 of Real Data Study 1 (= updating the

posteriors of the single-attribute parameters by the multiple-attribute item response data) improved the accuracy of the parameter estimation. Therefore, the estimation was conducted in two ways: (1) Using steps 1 and 2 and (2) using all steps 1, 2, and 3. Then, CCR, ASB, and RMSE were compared for both estimation results. RMSE and ASB can be defined respectively as:

$$\text{RMSE}=\sqrt{\frac{\sum_{j=1}^{N}(\pi_{jk}-\hat{\pi}_{jk})^2}{N}} \text{ and } \text{ASB}=\frac{\sum_{j=1}^{N}(\pi_{jk}-\hat{\pi}_{jk})}{N},$$

where N is the total number of examinees.

8.2 Results

8.2.1 Descriptive Statistics of True π_k and the Estimates ($\hat{\pi}_k$)

The descriptive statistics of the simulated (true) π_k –parameter, raw scores from the simulated item response data, and the parameter estimates ($\hat{\pi}_k$) for the four single attributes (A_1, A_2, A_3, A_4) were presented in Table 8.2. (for the first 20 and the last 10 examinees, the true π_{jk} and a_{jk}-parameters, and the raw scores were provided in Appendix C). In the table, $\hat{\pi}_k$ was reported separately for Step 1 & 2 (without updating single-attribute $\hat{\pi}_k$ by multiple-attribute data) and for Step 3 (with the updating).

The average true π_k of the four attributes ranged from .57 to 0.64 (M = .60) and the average $\hat{\pi}_k$ ranged from 0.56 to 0.62 (M = .58) in Step 1 & 2 and from 0.54 to 0.61 (M = .56) in Step 3. It was found that the average $\hat{\pi}_k$ in Step 1 & 2 was closer to the average true π_k and was slightly bigger than in Step 3.

8.2.2 Correct Classification Rate for Attribute Mastery

To evaluate the general accuracy of the attribute-mastery classification in BIBP, three measures of correct classification rate (CCR) were reported in Table 8.3: Marginal CCR for each attribute, CCR for the whole mastery pattern of individuals (all four attributes correct), and proportion of examinees having three or more attributes correctly classified. Also, the three measures were examined separately for Step 1 & 2 and Step 3 as shown in the table in order to find whether or not the updating procedure actually improved the accuracy of the parameter estimation.

Table 8.2 *Descriptive Statistics of True π_k , raw score, and the Estimate ($\hat{\pi}_k$) (N = 10000)*

	Parameter	*Min.*	*Max.*	*M*	*SD*
True	π_1	0.0100	0.9899	0.5830	0.2922
	π_2	0.0101	0.9900	0.5948	0.2900
	π_3	0.0101	0.9900	0.6367	0.2806
	π_4	0.0102	0.9899	0.5687	0.2915
Raw score		1	60	30.09 (50%)	13.89
Estimated in Step 1 & 2	$\hat{\pi}_1$	0.0625	0.9375	0.5751	0.2877
	$\hat{\pi}_2$	0.0625	0.9375	0.5834	0.2875
	$\hat{\pi}_3$	0.0625	0.9375	0.6155	0.2787
	$\hat{\pi}_4$	0.0625	0.9375	0.5629	0.2882
Estimated in Step 3	$\hat{\pi}_1$	0.0455	0.9545	0.5387	0.2500
	$\hat{\pi}_2$	0.0455	0.9545	0.5529	0.2531
	$\hat{\pi}_3$	0.0455	0.9545	0.6071	0.2682
	$\hat{\pi}_4$	0.0455	0.9545	0.5460	0.2693

Table 8.3 *Correct Classification Rate (CCR) for Attribute Mastery Patterns*

	Marginal CCR					*CCR for the whole pattern*	*CCR for three or more attributes*
	A_1	*A_2*	*A_3*	*A_4*	*M*		
Step 1 & 2	.825	.822	.798	.834	.820	.481	.834
Step 3	.770	.766	.820	.830	.796	.442	.794

As shown in Table 8.3, on average, 82% of the attributes were correctly classified in Step 1 & 2, which was 2.4% higher than in Step 3 (average marginal CCR = .796). Individually, proportion of the CCR for the whole four-attributes pattern was 48.1% and for the three or more attributes was 83.4% in Step 1 & 2, which were higher than in Step 3 by 1.9% and 4%, respectively. The result suggested that Step 3 was not effective in improving the accuracy of attribute-mastery classification. With respect to the marginal CCR for each attribute, A_4 showed the highest CCR (83.4%), A_1 and A_2 had 82.5% and 82.2%, respectively, and A_3 showed the lowest value (79.8%). It was an interesting finding that the order of CCR was equal to the order of attribute difficulty (p_k). That is, the harder attribute showed the better CCR; $p_4 = .44$, $p_1 = .47$, $p_2 = .49$, and $p_3 = .57$.

8.2.3 Accuracy of the Attribute-Mastery Probability Recovery

As a check of the estimation accuracy of the attribute-mastery probability (π_k), ASB and RMSE were obtained for each attribute in Table 8.4. Again, these measures were provided separately for Step 1&2 and Step 3 for the comparison purpose.

Table 8.4 *Average Signed Bias (ASB) and Root Mean Square Error (RMSE)*

		π_1	π_2	π_3	π_4	*M*
Step 1 & 2	*ASB*	.0112	.0126	.0152	.0079	.0117
	RMSE	.1382	.1354	.1337	.1397	.1368
Step 3	*ASB*	.0436	.0431	.0253	.0227	.0337
	RMSE	.1504	.1451	.1312	.1403	.1417

Mean ASB and RMSE of Step 1 & 2 were .0117 and .1368, respectively and they were smaller than those of Step 3 by .022 and .0049, respectively. The result suggested that the Step 3 (updating the single attribute parameters) did not improve the accuracy of

the π_k- parameter estimation. With respect to ASB for each attribute, interestingly, the order of its size was equal to the order of attribute difficulty as in the CCR result above. That is, the harder attribute showed the smaller ASB; .0079 for A_4, .0112 for A_1, 0126 for A_2, and .0152 for A_3.

8.3 Discussion

It was found that the marginal correct classification rate of the BIBP estimation was relatively high (about 80%) and the true π_k -parameter recovery error was small (ASB < .05, RMSE < .15). The BIBP parameter estimation was more accurate when no update was made for the single-attribute $\hat{\pi}_k$ by multiple-attribute data (Step 1&2). The finding suggests that the updating step (Step 3) does not actually improve the accuracy of the BIBP parameter estimation. In real data Study 1, it was discussed that the aberrant item response patterns (e.g., poor performance on either A_1 or A_2, but good performance on A_{12}) may be due to negative slips on A_1 (or A_2) or positive slips (successful guessing) on A_{12}. However, the updating step may not be helpful because it will increase the parameter-estimation accuracy in the negative-slip case but decrease the accuracy in the positive-slip case.

Another interesting finding was that more accurate true parameter recovery was found for harder attributes. The finding suggests that the attribute difficulty may affect the parameter estimation in BIBP. Therefore, the effect of attribute difficulty on the parameter estimation was evaluated in the next simulation study.

CHAPTER 9

SIMULATION STUDY 2: ACCURACY OF THE PARAMETER ESTIMATION UNDER VARIOUS CONDITIONS

In this study, the impact of the three variables on the BIBP parameter estimation was examined: (1) Attribute Correlation (2) Attribute Difficulty and (3) Sample Size. First, the correlations among the attributes in a test may vary depending on the characteristics of the attributes involved in the test. In this study, examinee attribute mastery patterns were generated to have various degrees of inter-attribute correlations: (a) zero, (b) low, (c) medium, and (d) high. Second, in Simulation Study 1, it was found that the BIBP parameter estimation was more accurate (i.e., higher CCR, lower ASB) for the harder attributes. Therefore, in this study, different levels of attribute difficulty were simulated in the item design; easy (p_k = .7 to .9), medium (p_k = .4 to .6), and hard (p_k = .1 to .3). Third, three different sample sizes (100, 300, and 500) were used to examine if sample size affects the accuracy of parameter estimation and if the sample size also interacts with the other two factors (attribute correlation, attribute difficulty). For the simulation, Excel VBA (macro) and the R program were used. It was elaborated how to simulate the various attribute correlations and the difficulties below.

9.1 Method

9.1.1 Attribute Correlation and Attribute Difficulty

The same simulated item design as in Study 1 was used in which 60 items measured four attributes (see Table 8.1). In order to simulate different types of relations of four attributes, a multivariate normal distribution with a mean vector of zeros and one

of the following correlation matrices was used to generate $z_k \sim MVN(\mathbf{0}, \boldsymbol{R})$: $\boldsymbol{R_1}$ (no correlations), $\boldsymbol{R_2}$ (low correlations), $\boldsymbol{R_3}$ (medium correlations), and $\boldsymbol{R_4}$ (high correlations),

$$\boldsymbol{R_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{R_2} = \begin{bmatrix} 1 & .30 & .20 & .10 \\ .30 & 1 & .20 & .20 \\ .20 & .20 & 1 & .30 \\ .10 & .20 & .30 & 1 \end{bmatrix}, \boldsymbol{R_3} = \begin{bmatrix} 1 & .60 & .50 & .40 \\ .60 & 1 & .50 & .50 \\ .50 & .50 & 1 & .60 \\ .40 & .50 & .60 & 1 \end{bmatrix}, \text{ and}$$

$$\boldsymbol{R_4} = \begin{bmatrix} 1 & .90 & .80 & .70 \\ .90 & 1 & .80 & .80 \\ .80 & .80 & 1 & .90 \\ .70 & .80 & .90 & 1 \end{bmatrix}.$$

As in Simulation Study 1 (see Chapter 8.1.2), π_{jk} (for examinee *j* and attribute *k*) was generated by comparing the z_{jk} score generated above with *INVNORM* (p_k):

$$\begin{cases} \pi_{jk} \sim U(.70, .99) \text{ and } a_{jk} = 1, & \text{if } z_k \leq INVNORM(p_k) \\ \pi_{jk} \sim U(.01, .69) \text{ and } a_{jk} = 0, & \text{otherwise} \end{cases}$$

$$\text{where } p_k \sim \begin{cases} U(.10, .30) \text{ for hard attribute} \\ U(.40, .60) \text{ for medium} - \text{difficulty attribute} \\ U(.70, .90) \text{ for easy attribute} \end{cases}$$

To simulate different levels of attribute difficulties, p_k (= probability that attribute *k* is mastered in all examinees) was randomly drawn from *U* (.10, .30) for hard, *U* (.40, .60) for medium, and *U* (.70, .90) for easy attributes, respectively. As mentioned in Simulation Study 1, *INVNORM* (p_k) is a z-score corresponding to the cumulative probability of p_k under standard normal distribution. If z_{jk} is smaller than or equal to *INVNORM* (p_k), then, examinee *j* is considered to have mastered the attribute, otherwise, the examinee does not. For hard attributes [$p_k \sim U$ (.10, .30)], examinees are less likely to show the mastery than for easy attributes [$p_k \sim U$ (.70, .90)]. α_{jk} was also generated based on π_{jk} (mastery if $\pi_{jk} \geq 0.7$; non-mastery otherwise) as in Simulation Study 1.

In addition, the unidimensionality was checked for the four different attribute correlations. It was hypothesized that the data with higher inter-attribute correlations would fit better the unidimensional IRT model. 3PLM was adopted for the unidimensionality check. IRTPRO (item response theory for patient-reported outcomes; Thissen, 2010) program was used for the IRT analysis.

9.1.2 Sample Size

Henson et al. (2006) observed that higher attribute correlations were associated with slightly higher CCR's in their simulation study. However, no simulation study that links sample size with attribute correlation or attribute difficulty was found for the CDA parameter estimation. Therefore, sample size was added to this simulation study as the third conditional variable to see whether it interacts with attribute correlation and difficulty in the parameter estimation.

Table 9.1 *Three Simulation Variables and their Levels*

Attribute Correlation (4)	*Attribute Difficulty (3)*	*Sample Size (3)*	Simulated Conditions (36)
$\mathbf{R_1}$ (no correlation)	Easy	100	4×3×3=36 *(replication=100)*
$\mathbf{R_2}$ (low correlations)	Medium-difficulty	300	
$\mathbf{R_3}$ (medium correlations)	Hard	500	
$\mathbf{R_4}$ **(**high correlations)			

In this study, the effect of three different sample sizes on parameter estimation was examined for each level of the attribute correlation and attribute difficulty: 100

subjects (small sample size), 300 subjects (medium sample size), and 500 subjects (relatively large sample size). Therefore, there were a total of 36 simulated conditions (4 attribute correlations × 3 attribute difficulties × 3 sample sizes) as presented in Table 9.1. Every simulated condition was replicated 100 times.

9.1.3 Item Response Data Generation

As in Simulation Study 1, binary response (x_{ij}) to item *i* was generated for examinee *j* using a *Uniform* distribution, *U* (0,1) based on π_{jk} generated above:

$$x_{ij} = \begin{cases} 1\ (= correct) & if\ U(0,1) \leq \pi_{jk}^{*} \\ 0\ (= incorrect) & otherwise \end{cases},$$

where $\pi_{jk}^{*} = \pi_{jk}$ for single attribute-items, or the smallest of all involved π_{jk}s for multiple-attributes items.

In each of the 36 conditions as presented in Table 9.1, item response vector to 60 items was generated. For each condition, the data generation was replicated 100 times. Then, the parameters, π_{jk} and α_{jk}, were estimated assuming they were unknown in the same way as in Real Data Study 1. CCR, ASB, and RMSE were used to check the parameter-recovery accuracy for α_{jk} and π_{jk}.

9.1.4 Comparison with the DINA estimation

The BIBP estimation was compared with the DINA estimation in the parameter-recovery accuracy. For the comparison, the item response data was regenerated for each of the 36 simulation conditions (replication = 20), and the CCR, ASB, and RMSE were compared between the DINA and BIBP estimations using the same data. The impact of the three variables (Attribute Correlation, Attribute Difficulty and Sample Size) for the BIBP parameter estimation was also examined and compared with the BIBP estimation.

9.2 Results

9.2.1 Attribute Correlation

Marginal means of CCR, ASB, and RMSE were presented for each level of the simulated attribute correlation in Table 9.2. Low and medium correlations showed slightly higher CCR's than no and high correlations. However, ANOVA test indicated that there was no significant difference for the different levels of attribute correlations in all the three measures; CCR [$F(3,3596) = 2.19$, $p = .087$, $\eta_p^2 = .002$], ASB[$F(3,3596) = 2.04$, $p = .106$, $\eta_p^2 = .002$], and RMSE[$F(3,3596) = 1.31$, $p = .271$, $\eta_p^2 = .001$].

Table 9.2 *CCR, ASB, and RMSE for Attribute Correlation*

Simulation Variable	*Levels*	*CCR*	*ASB*	*RMSE*
Attribute Correlation	No correlation	.808	0.015	0.135
	Low	.814	0.014	0.136
	Medium	.815	0.013	0.136
	High	.811	0.015	0.135
	Total	0.812	0.014	0.136

9.2.2 Attribute Difficulty

Attribute difficulty showed a significant effect on the BIBP parameter estimation. For all the three measures, CCR, ASB, and RMSE, significant differences were found among the three difficulty levels; CCR [$F(2,3597) = 22103.94$, $p = .000$, $\eta_p^2 = .93$], ASB[$F(2,3597) = 13332.99$, $p = .000$, $\eta_p^2 = .88$], and RMSE[$F(2,3597) = 3498.99$, $p = .000$, $\eta_p^2 = .66$]. The large η_p^2 indicated that most of the total variances in CCR (93%), ASB (88%), and RMSE (66%) were explained by attribute difficulty. As shown in Table 9.3, and Figure 9.1, the harder attribute resulted in higher CCR but larger RMSE on average. It was speculated that the observation was due to the following reason: In

general, more false negative classifications (classifying mastery as non-mastery) were observed than false positive classifications (classifying a non-mastery as a mastery) because of the $\hat{\pi}_k$ range difference between mastery and non-mastery. In other words, the range of $\hat{\pi}_k$ for mastery (.70 to .99) is smaller than the $\hat{\pi}_k$-range for non-mastery (.01 to .69) based on the definition in the current study. Therefore, the probability of false negative classification is higher than that of false positive classification. Note that if the attribute difficulty increased (=harder), then the number of examinees who mastered the attribute decreased, thus making a smaller number of false negative classifications, which resulted in higher CCR and larger RMSE for harder attributes.

Table 9.3 *CCR, ASB, and RMSE for Attribute Difficulty*

Simulation Variable	*p_k-range*	*CCR*	*ASB*	*RMSE*
Attribute	.68 ~ .88 (easy)	.734	0.033	0.129
Difficulty	.44 ~ .55 (medium)	.813	0.014	0.135
	.11~ .30 (hard)	.890	-0.005	0.142
	Total	0.812	0.014	0.136

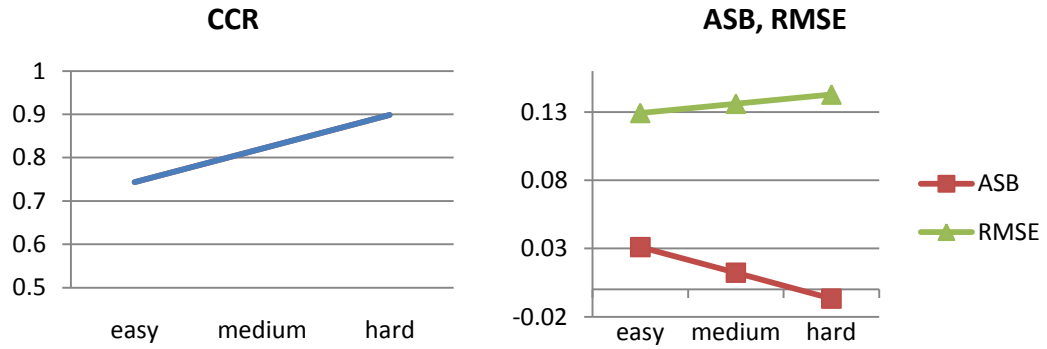


Figure 9.1 Effect of Attribute Difficulty on CCR, ASB, and MRSE

Post-hoc tests (Tukey, Scheffe, Bonferroni) indicated that the three levels of difficulties (easy, medium, hard) was significantly different from each other in all the three measures (CCR, ASB, and RMSE).

9.2.3 Sample Size

A significant effect of sample size was also found for the three measures; CCR [$F(2,3597) = 11.44$, $p = .000$, $\eta_p^2 = .006$], ASB[$F(2,3597) = 12.38$, $p = .000$, $\eta_p^2 = .007$], and RMSE[$F(2,3597) = 8.14$, $p = .000$, $\eta_p^2 = .005$]. However, effect sizes (η_p^2) for the three measures were very small (less than .01), which suggested that sample size was not a strong factor as attribute difficulty on the BIBP parameter estimation. Post-hoc tests (Tukey, Scheffe, Bonferroni) indicated that the significant difference existed between the sample size of 500 and the other sample sizes (100 and 300) and no significant difference existed between the sample sizes of 100 and 300 for all the three measures (CCR, ASB, and RMSE). Table 9.4 and Figure 9.2 presented the different marginal means of CCR, ASB, and RMSE for each sample size.

Table 9.4 *CCR, ASB, and RMSE for Sample Size*

Simulation Variable	*Levels*	*CCR*	*ASB*	*RMSE*
Sample Size	100	.808	0.015	0.135
	300	.809	0.015	0.135
	500	.820	0.012	0.136
	Total	0.812	0.014	0.136

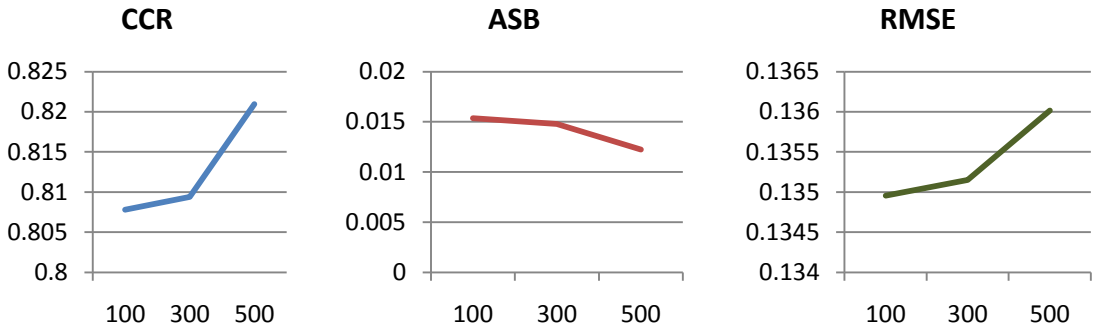


Figure 9.2 Effect of Sample Size on CCR, ASB, and MRSE

9.2.4 Interactions of the Simulation Variables

To check the interactions and the main effects (unique effect) of the three simulation variables after controlling for the all the other effects, a three-way ANOVA (Attribute Correlation × Attribute Difficulty × Sample Size) was conducted. For the 36 simulation conditions (4 Attribute Correlations × 3 Attribute Difficulties × 3 Sample Sizes), average CCR, ASB, and RMSE were also provided in Appendix D.

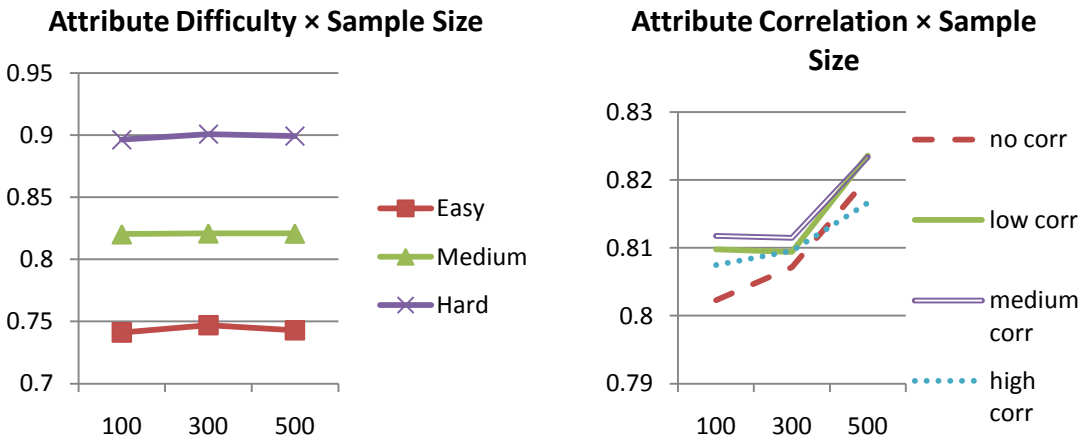


Figure 9.3 Two-Way Interactions of Attribute Difficulty × Sample Size and of Attribute Correlation × Sample Size in CCR

For CCR, significant 2-way interactions between Attribute Difficulty and Sample Size [F (2,3564) = 6.44, p = .000, η_p^2 = .007] and between Attribute Correlation and Sample Size [F (2,3564) = 3.68, p = .001, η_p^2 = .006] and a significant 3-way interaction [F (2,3564) = 4.85, p = .000, η_p^2 = .016] were found. However, their effect sizes were very small for the 2-way interactions (η_p^2 <.01) and small for the 3-way interaction.

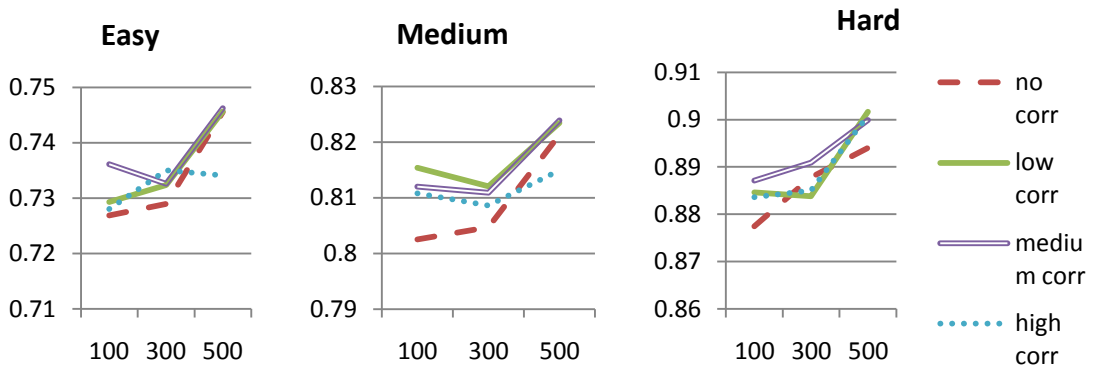


Figure 9.4 Attribute Difficulty × Sample Size for Attribute Difficulties (3-Way Interaction) in CCR

As presented in Figure 9.3, the Sample Size's effect was slightly different for easy attributes (smaller CCR for 500 sample-size than 300) compared with medium difficulty attributes. Figure 9.3 and 9.4 showed that Sample Size's effect on CCR was different for high attribute correlations compared to the other levels of correlations and the overall pattern of the Sample Size effect varied for different Attribute Difficulties (hard attributes showed more linear patterns of Sample Size effect). The main effect of Attribute Correlation (unique effect) after controlling for all the other effects was found significant, F (2,3564) = 33.43, p = .000, η_p^2 = .027, while the other two main effects (Attribute Difficulty, Sample Size) became more significant.

For ASB, only the 2-way interaction between Attribute Difficulty and Sample Size and the 3-way interaction were significant, F (2,3564) = 6.15, p = .000, η_p^2 = .007 and F (2,3564) = 4.29, p = .000, η_p^2 = .014, respectively. However, for RMSE, no significant interaction was found. On both ASB and RMSE, the unique effect of Attribute Correlation was found significant, F (2,3564) = 18.81, p = .000, η_p^2 = .016 and F (2,3564) = 3.92, p = .008, η_p^2 = .003l while the main effects of the other two factors remained significant after holding all the other effects.

9.2.5 Comparison with the DINA Estimation

To compare with the DINA estimation, the item response data was regenerated for each of the 36 simulation conditions (replication = 20), and the average CCR, ASB, and RMSE were compared between the DINA and BIBP estimations using the same data. Marginal means and standard deviations of CCR, ASB, and RMSE of each level of the three simulation variables were provided for the two models in Table 9.5.

For the 36 simulation conditions (4 Attribute Correlations × 3 Attribute Difficulties × 3 Sample Sizes), average CCR, ASB, and RMSE for the DINA estimation were also provided in Appendix E.

The overall CCR of DINA (=.867) was higher than that of BIBP (=.786) while the overall ASB and RMSE of BIBP (.015 and .140, respectively) were smaller than those of DINA (-.051 and .271, respectively). The standard deviations of the three estimation-accuracy measures were larger in the DINA estimation than in the BIBP estimation. Generally, the less accurate estimation (lower CCR, bigger ASB/RMSE) showed the larger standard deviation of the measure.

Table 9.5 *Marginal Means and Standard Deviations of CCR, RMSE, and ASB for the DINA and BIBP Estimations*

Simulation Variables	*Levels*		*DINA*			*BIBP*		
			CCR	*ASB*	*RMSE*	*CCR*	*ASB*	*RMSE*
Attribute Correlation	No corr.	*M*	0.853	-0.066	0.269	0.772	0.018	0.139
		SD	0.118	0.073	0.048	0.091	0.018	0.008
	Low Corr.	*M*	0.829	-0.075	0.284	0.788	0.014	0.140
		SD	0.133	0.088	0.066	0.084	0.017	0.009
	Med Corr.	*M*	0.872	-0.046	0.268	0.794	0.014	0.140
		SD	0.102	0.086	0.044	0.085	0.016	0.008
	High Corr.	*M*	0.913	-0.017	0.264	0.789	0.014	0.140
		SD	0.054	0.092	0.034	0.080	0.016	0.007
Attribute Difficulty	p_k =.74~.92 (easy)	*M*	0.946	-0.127	0.232	0.694	0.032	0.133
		SD	0.043	0.023	0.033	0.023	0.006	0.005
	p_k =.54~.67 (medium)	*M*	0.888	-0.022	0.270	0.774	0.017	0.138
		SD	0.059	0.040	0.035	0.040	0.009	0.006
	p_k =.14~.34 (hard)	*M*	0.765	-0.003	0.313	0.889	-0.004	0.147
		SD	0.118	0.110	0.043	0.021	0.007	0.006
Sample Size	100	*M*	0.833	-0.081	0.293	0.781	0.016	0.139
		SD	0.145	0.099	0.062	0.088	0.018	0.009
	300	*M*	0.887	-0.035	0.260	0.783	0.015	0.139
		SD	0.077	0.079	0.035	0.085	0.016	0.007
	500	*M*	0.880	-0.037	0.262	0.793	0.014	0.140
		SD	0.088	0.076	0.041	0.082	0.016	0.007
Total		*M*	0.867	-0.051	0.271	0.786	0.015	0.140
		SD	0.110	0.088	0.050	0.085	0.017	0.008

In the DINA estimation, all the main effects of the three variables (Attribute Correlation, Attribute Difficulty, and Sample Size) were significant at $\alpha = .01$ on the three measures (CCR, ASB, RMSE). For Attribute Correlation, more accurate parameter estimation (i.e., higher CCR, smaller ASB/RMSE) was observed in the higher correlations except in low correlation which showed the least accuracy in parameter estimation (see Figure 9.5). In addition, for the different correlations, the data-model fit for a unidimensional IRT model (3PLM) was compared using -2loglikelihood which indicates the degree of departure of the data from the model. To get the simple effect of

the correlation, a random data with only medium-difficulty and 300-sample size was chosen for each of the four correlations. As expected, the higher correlation-data showed the better fit as presented in Table 9.6, which suggested that the higher inter-attribute correlations, the stronger unidimensioanlity the data had.

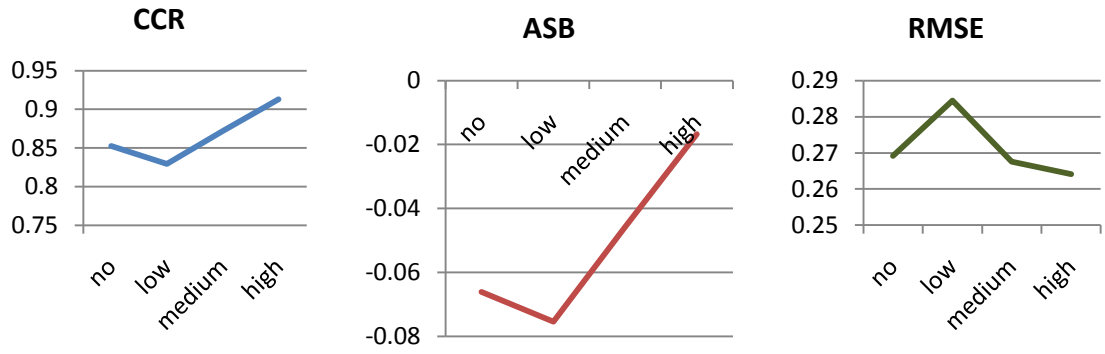


Figure 9.5 Effect of Attribute Correlation on CCR, ASB, and MRSE in DINA

Table 9.6 *Unidimensional IRT Model (3PLM) Fit for the Four Correlations*

Model fit	*No Corr.*	*Low Corr.*	*Medium Corr.*	*High Corr.*
-2loglikelihood	22859.27	22465.44	21961.14	20984.72

For Attribute Difficulty, the easier attributes were estimated more accurately than the harder attributes, which was opposite to the BIBP estimation where the harder, the better estimations were observed (see Figure 9.6). Note that the larger standard deviations were observed in the DINA estimation of hard attributes. Post-hoc tests (Tukey, Scheffe, Bonferroni) indicated that every difficulty level was significantly different from each other in the three measures.

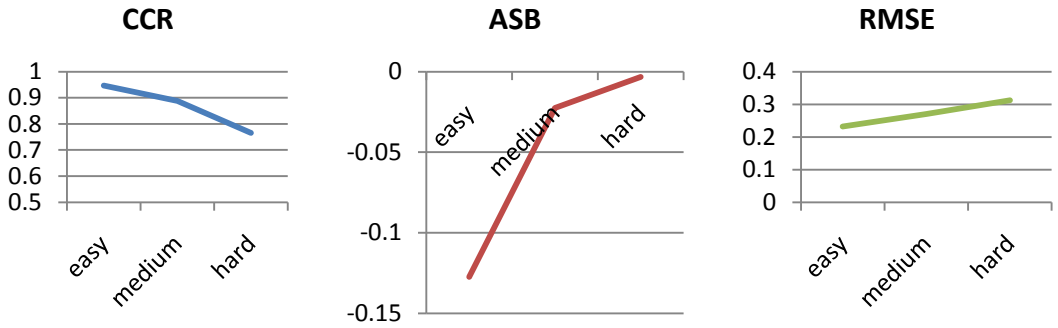


Figure 9.6 Effect of Attribute Difficulty on CCR, ASB, and MRSE in DINA

For Sample Size, more accurate estimation was observed in the sample size of 300 or 500 than the sample size of 100 (see Figure 9.7). The post-hoc tests showed a significant difference between the sample sizes of 100 and 300 (or 500) but no significant difference between the sample sizes of 300 and 500. In addition, all the interactions of the three simulation variables (Attribute Correlation, Attribute Difficulty, Sample Size) were found significant at the .05 level on CCR, ASB, and RMSE.

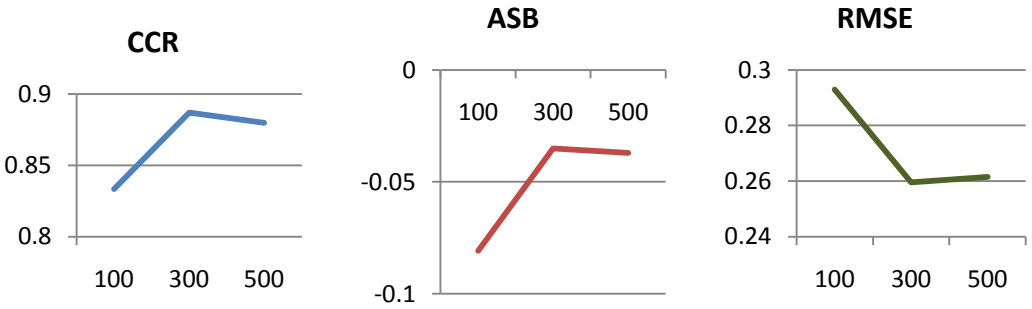


Figure 9.7 Effect of Sample Size on CCR, ASB, and MRSE in DINA

9.3 Discussion

The overall accuracy of the BIBP parameter estimation of this study (CCR = .812, ASB = .014, RMSE = .136) was similar to the general accuracy result of Study 1 (CCR = .820, ASB = .012, RMSE = .137). The DINA estimation showed higher overall CCR (= .867) but the bigger overall biases and estimation errors (ASB = -.051, RMSE = .271) than the BIBP estimation. The three simulation variables (Attribute Correlation, Attribute Difficulty, and Sample Size) showed significant impacts on the parameter estimations of both models. However, they affected differently the two models.

In the BIBP estimation, Attribute Difficulty was the strongest factor and explained most variance in the correct classification (η_p^2 = .934) although the other factors (Attribute Correlation and Sample size) were significant. The harder the attribute was, the more accurate its parameter estimation was and the low and medium attribute correlation resulted in higher accuracy in the parameter estimation. Also, 500-sample size showed the higher estimation accuracy than 100-or 300-sample size. Therefore, the condition of hard attribute difficulty, low attribute correlation, and sample size of 500 showed the highest CCR (= .902) while the condition of easy attribute difficulty, no attribute correlation, and sample size of 100 had the lowest CCR (= .727).

In the DINA estimation, Attribute Correlation showed a strong effect size on the correct classification (η_p^2 = .211) although Attribute Difficulty remained as the strongest factor (η_p^2 = .618). However, unlike the BIBP estimation, the highly correlated and easy attributes showed the highest accuracy of the parameter estimation. Therefore, the condition of easy attribute difficulty, high attribute correlation, and sample size of 500

showed the highest CCR (= .968) while the condition of hard attribute difficulty, low attribute correlation, and sample size of 100 had the lowest CCR (= .582).

CHAPTER 10

CONCLUSION

Cognitive diagnostic assessment (CDA) is a new theoretical framework for psychological and educational testing that is designed to diagnose detailed information about examinees' mastery of attributes in a test (specific knowledge structures and processing skills). Cognitive psychology plays a key role in CDA since it provides the basis to identify and understand the component processing skills or attributes underlying the test performance. Also, a well-developed cognitive theory can provide a framework for guiding item selection and design (i.e., cognitively based item generation).

During the last three decades, more than a dozen psychometric models have been developed for CDA. Although they have successfully provided useful diagnostic information about the examinee, most CDMs are complex due to a large number of parameters in proportion to the number of attributes (= k) to be measured in an item. For example, fusion model has k+2 parameters, the DINO model includes 2k parameters, and LCDM has even 2^k parameters. The large number of parameters causes heavy computational demands for the estimation. For some CDM's (e.g., Fusion, LCDM), MCMC algorithm is used for the parameter estimation because it is easier to extend to parametrically complex models than Expectation Maximization (EM) algorithms. However, the MCMC causes even heavier computational demand than the marginal maximum likelihood estimation (MMLE) with the EM algorithm. It takes several hours even for a single estimation and a day or more for more complex models or large amounts of data. Also, the MCMC can be misused easily because of the complexity of its

algorithms. Another issue in CDA is that a variety of specific software applications were developed for the chosen CDMs and most of them are not user-friendly.

Therefore, purpose of this study was to propose a simple and effective method for CDA without heavy computational demand and using a user-friendly software application. Bayesian inference for binomial proportion was applied to CDA because of the following reason: When we have binomial observations such as item responses (right/wrong), using a *beta* distribution as a prior of a parameter to estimate (i.e., attribute-mastery probability) makes it very simple to find the *beta* posterior of the parameter without any integration. However, the application of BIBP to CDA can be flexible depending on the test item-attribute design and examinees' attribute-mastery patterns. In this study, effective ways of applying the BIBP method was explored using real data studies and simulation studies. Also, other preexisting diagnosis models such as DINA and LCDM were compared to the BIBP method in their diagnosis results.

In the real data studies, the BIBP method was applied to a mathematical test data which involved different total number of attributes based on Q-matrics: four attributes (based on four mathematical standards) or ten attributes (based on ten benchmarks of the four standards). Also, the BIBP method was compared with DINA and LCDM in diagnosing examinees' attributes- mastery using the same data set (four attributes). There were slight differences in the attribute mastery probability estimate ($\hat{\pi}_k$) among the three model (DINA, LCDM, BIBP), which could result in different attribute mastery pattern (α_k)- diagnosis results. Interestingly, the diagnosis results of BIBP and LCDM were very similar while the DINA result was quite different from them. In both four-attribute and ten-attribute data studies, aberrant item response patterns such as poor performance on

single-attribute items (e.g., A_1, A_2) and better performance on multiple attribute items (A_{12}) was found. It was speculated that negative slips on the single-attribute items or positive slips (guessing) on the multiple-attribute items could result in the aberrant result. Also, the aberrant response pattern may suggest that there was a compensatory nature between the attributes (e.g., A_1 or A_2) in a combined multiple attribute (e.g., A_{12}).

Simulation studies were conducted to (1) evaluate general accuracy of the BIBP parameter estimation and the effectiveness of updating $\hat{\pi}_k$ of single attributes by $\hat{\pi}_k$ of multiple attributes , (2) examinee the impact of various factors such as attribute correlation (no, low, medium, and high), attribute difficulty (easy, medium, and hard) and sample size (100, 300, and 500) on the consistency of the parameter estimation of BIBP, and (3) compare the BIBP method with the DINA model in the accuracy of recovering true parameters. It was found that the general accuracy of the BIBP method in the true parameter estimation was relatively high and the DINA estimation showed higher overall CCR but the bigger overall biases and estimation errors than the BIBP estimation. It was also found that the BIBP parameter estimation was more accurate when no update was made for the single-attribute $\hat{\pi}_k$ by multiple-attribute data. Therefore, it appears that the updating step does not actually improve the accuracy of the BIBP parameter estimation. The updating step can be helpful for the case of negative-slips on single-attribute items but it will decrease the parameter estimation accuracy for the case of positive-slips on multiple-attributes items.

The three simulation variables (Attribute Correlation, Attribute Difficulty, and Sample Size) showed significant impacts on the parameter estimations of both models. However, they affected differently the two models. In the BIBP estimation, Attribute

Difficulty was the strongest factor and explained most variance (93.4%) in the correct classification and the harder attributes showed the more accurate classification. In the DINA estimation, although Attribute Difficulty still accounted for majority of the variance in the correct classification (62%), the effect of Attribute Correlation became stronger, explaining 21% of variance of the correct classification. However, unlike the BIBP estimation, the highly correlated and easy attributes showed the highest accuracy of the parameter estimation. In addition, for the different correlations, the data-model fit for a unidimensional IRT model (3PLM) was compared. As expected, the higher correlation-data showed the stronger unidimensioanlity.

One big advantage of the BIBP method over other CDM's was the fast parameter estimation. For the four-attribute data, it took about a second to estimate $\hat{\pi}_k$ and α_k for all the 2993 examinees while it took about 30 minutes by DINA and about one and a half hours by LCDM using the Pentium(R) dual-core CPU (1.50 GHz). It was because the BIBP estimation needed no iteration procedure such as MMLE-EM or MCMC unlike the DINA and LCDM estimation. Especially for LCDM, although the number of parameters per item is 16 for the current four-attribute data, it will be doubled as one attribute is added. Therefore, for ten-attribute data, it will become 16 x 2^6 (=1024) parameters per item, which will greatly increase the computational demand and the estimation time. Also, the BIBP parameter estimation is available for a user-friendly program such as Excel. Therefore, the BIBP method may benefit general users such as school teachers who administer classroom tests on a daily base. Another advantage of the BIBP method was to provide the estimates of both single-attribute parameters (e.g., π_1, π_2, π_3) and multiple-(or combined) attribute parameters (e.g., π_{12}, π_{23}, π_{123}), which would be helpful in

diagnosing examinees' ability, especially when the attributes within a test items are compensatory.

In conclusion, the application of BIBP appears an effective method for CDA with a relatively high accuracy of diagnosing examinees' attribute mastery. However, in the future, it needs to be further explored how to infer the parameters of the unmeasured attributes directly by test items as in Real Data Study 2. Also, R-programming can be used for the BIBP parameter estimation in the future study. An increased interest in the research about the combination of CDA and computerized testing (e.g., Tatsuoka & Tatsuoka, 1997; McGlohen, 2004) has already occurred. However, few studies dealing with response time within the CDA framework have been found. Therefore, the potential benefits which response time data can give for the CDA application need to be explored in future research.

APPENDIX A: FOUR STANDARDS AND THEIR BENCHMARKS

Standard	*Benchmark*
1. Number and Computation	1) Number Sense: The student demonstrates number sense for real numbers and simple algebraic expressions in a variety of situations. 2) Number System and Their Properties: The student demonstrates an understanding of the real number system; recognizes, applies, and explains their properties; and extends these properties to algebraic expressions. 3) Computation: The student models, performs, and explains computation with rational numbers, the irrational number pi, and algebraic expressions in a variety of situations.
2. Algebra	4) Variable, Equations, and Inequalities: The student uses variables, symbols, real numbers, and algebraic expressions to solve equations and inequalities in a variety of situations. 5) Functions: The student recognizes, describes, and analyzes constant, linear, and nonlinear relationships in a variety of situations. 6) Models: The student generates and uses mathematical models to represent and justify mathematical relationships found in a variety of situations.
3. Geometry	7) Geometric Figures and Their Properties: The student recognizes geometric figures and compares their properties in a variety of situations. 8) Geometry from an Algebraic Perspective: The student uses an algebraic perspective to examine the geometry of two dimensional figures in a variety of situations.
4. Data	9) Probability: The student applies the concepts of probability to draw conclusions, generate convincing arguments, and make predictions and decisions including the use of concrete objects in a variety of situations. 10) Statistics: The student collects, organizes, displays, explains, and interprets numerical (rational) and non-numerical data sets in a variety of situations.

APPENDIX B

FOUR-ATTRIBUTE DATA ESTIMATION RESULTS

(REAL DATA STUDY 1)

Table B.1 *Examinee Attribute Mastery Probability Estimates (Step 1&2)*

ID	Raw score	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
1	77	0.88	0.85	0.81	0.96	0.90	0.82	0.88	0.84	0.82
2	79	0.88	0.91	0.97	0.88	0.84	0.92	0.88	0.91	0.87
3	79	0.88	0.85	0.81	0.96	0.90	0.82	0.88	0.89	0.86
4	53	0.71	0.68	0.58	0.29	0.66	0.61	0.35	0.38	0.45
5	68	0.88	0.44	0.92	0.54	0.69	0.47	0.58	0.52	0.60
6	77	0.79	0.91	0.92	0.71	0.84	0.92	0.73	0.79	0.82
7	67	0.54	0.85	0.97	0.63	0.64	0.81	0.58	0.56	0.71
8	37	0.38	0.26	0.36	0.38	0.46	0.25	0.42	0.30	0.40
9	75	0.88	0.85	0.92	0.79	0.81	0.86	0.81	0.85	0.82
10	41	0.54	0.44	0.47	0.46	0.46	0.42	0.50	0.48	0.44
11	68	0.54	0.79	0.86	0.63	0.71	0.81	0.58	0.68	0.66
12	36	0.54	0.26	0.47	0.46	0.31	0.25	0.5	0.3	0.35
13	63	0.79	0.5	0.75	0.88	0.63	0.47	0.73	0.61	0.56
14	61	0.54	0.79	0.58	0.54	0.74	0.61	0.58	0.62	0.61
15	69	0.63	0.62	0.86	0.96	0.72	0.64	0.65	0.66	0.69
16	42	0.38	0.62	0.31	0.54	0.47	0.34	0.42	0.38	0.38
17	66	0.96	0.62	0.64	0.63	0.75	0.64	0.65	0.7	0.65
18	62	0.71	0.62	0.58	0.63	0.72	0.61	0.65	0.7	0.66
19	77	0.88	0.79	0.92	0.79	0.84	0.81	0.81	0.85	0.87
20	63	0.63	0.79	0.81	0.54	0.67	0.75	0.58	0.68	0.61
:	**:**	:	:	:	:	:	:	:	:	:
2984	27	0.21	0.21	0.47	0.38	0.25	0.19	0.27	0.25	0.23
2985	73	0.79	0.74	0.86	0.96	0.78	0.75	0.81	0.75	0.77
2986	46	0.46	0.62	0.69	0.46	0.43	0.58	0.42	0.38	0.61
2987	50	0.63	0.62	0.47	0.54	0.63	0.45	0.58	0.56	0.50
2988	43	0.29	0.56	0.47	0.46	0.50	0.45	0.35	0.26	0.45
2989	70	0.71	0.85	0.97	0.54	0.74	0.86	0.58	0.62	0.66
2990	71	0.88	0.79	0.92	0.54	0.84	0.81	0.58	0.62	0.61
2991	52	0.63	0.50	0.69	0.71	0.43	0.53	0.65	0.57	0.56
2992	82	0.71	0.91	0.97	0.96	0.88	0.92	0.73	0.79	0.82
2993	67	0.71	0.68	0.86	0.71	0.78	0.64	0.73	0.70	0.69

Table B.2 *Examinee Attribute Mastery Patterns (Step 1&2)*

ID	Raw score	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
1	77	1	1	1	1	1	1	1	1	1
2	79	1	1	1	1	1	1	1	1	1
3	79	1	1	1	1	1	1	1	1	1
4	53	1	0	0	0	0	0	0	0	0
5	68	1	0	1	0	0	0	0	0	0
6	77	1	1	1	1	1	1	1	1	1
7	67	0	1	1	0	0	1	0	0	1
8	37	0	0	0	0	0	0	0	0	0
9	75	1	1	1	1	1	1	1	1	1
10	41	0	0	0	0	0	0	0	0	0
11	68	0	1	1	0	1	1	0	0	0
12	36	0	0	0	0	0	0	0	0	0
13	63	1	0	1	1	0	0	1	0	0
14	61	0	1	0	0	1	0	0	0	0
15	69	0	0	1	1	1	0	0	0	0
16	42	0	0	0	0	0	0	0	0	0
17	66	1	0	0	0	1	0	0	1	0
18	62	1	0	0	0	1	0	0	1	0
19	77	1	1	1	1	1	1	1	1	1
20	63	0	1	1	0	0	1	0	0	0
:	**:**	**:**	**:**	**:**	**:**	**:**	**:**	**:**	**:**	**:**
2984	27	0	0	0	0	0	0	0	0	0
2985	73	1	1	1	1	1	1	1	1	1
2986	46	0	0	0	0	0	0	0	0	0
2987	50	0	0	0	0	0	0	0	0	0
2988	43	0	0	0	0	0	0	0	0	0
2989	70	1	1	1	0	1	1	0	0	0
2990	71	1	1	1	0	1	1	0	0	0
2991	52	0	0	0	1	0	0	0	0	0
2992	82	1	1	1	1	1	1	1	1	1
2993	67	1	0	1	1	1	0	1	1	0

Table B.3 *Examinee Attribute Mastery Probability Estimates (Step 3)*

ID	Raw score	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
1	77	0.88	0.85	0.81	0.96	0.9	0.82	0.88	0.84	0.82
2	79	0.88	0.91	0.97	0.88	0.84	0.92	0.88	0.91	0.868
3	79	0.88	0.85	0.81	0.96	0.9	0.82	0.88	0.89	0.86
4	53	0.71	0.68	0.58	0.29	0.66	0.61	0.35	0.38	0.447
5	68	0.88	0.44	0.92	0.54	0.69	0.47	0.58	0.52	0.604
6	77	0.79	0.91	0.92	0.71	0.84	0.92	0.73	0.79	0.816
7	67	0.54	0.85	0.97	0.63	0.64	0.81	0.58	0.56	0.711
8	37	0.38	0.26	0.36	0.38	0.46	0.25	0.42	0.3	0.396
9	75	0.88	0.85	0.92	0.79	0.81	0.86	0.81	0.85	0.816
10	41	0.54	0.44	0.47	0.46	0.46	0.42	0.5	0.48	0.438
11	68	0.54	0.79	0.86	0.63	0.71	0.81	0.58	0.68	0.658
12	36	0.54	0.26	0.47	0.46	0.31	0.25	0.5	0.3	0.354
13	63	0.79	0.5	0.75	0.88	0.63	0.47	0.73	0.61	0.563
14	61	0.54	0.79	0.58	0.54	0.74	0.61	0.58	0.62	0.605
15	69	0.63	0.62	0.86	0.96	0.72	0.64	0.65	0.66	0.688
16	42	0.38	0.62	0.31	0.54	0.47	0.34	0.42	0.38	0.38
17	66	0.96	0.62	0.64	0.63	0.75	0.64	0.65	0.7	0.646
18	62	0.71	0.62	0.58	0.63	0.72	0.61	0.65	0.7	0.66
19	77	0.88	0.79	0.92	0.79	0.84	0.81	0.81	0.85	0.868
20	63	0.63	0.79	0.81	0.54	0.67	0.75	0.58	0.68	0.605
:	**:**	:	:	:	:	:	:	:	:	:
2984	27	0.21	0.21	0.47	0.38	0.25	0.19	0.27	0.25	0.229
2985	73	0.79	0.74	0.86	0.96	0.78	0.75	0.81	0.75	0.771
2986	46	0.46	0.62	0.69	0.46	0.43	0.58	0.42	0.38	0.605
2987	50	0.63	0.62	0.47	0.54	0.63	0.45	0.58	0.56	0.5
2988	43	0.29	0.56	0.47	0.46	0.5	0.45	0.35	0.26	0.447
2989	70	0.71	0.85	0.97	0.54	0.74	0.86	0.58	0.62	0.658
2990	71	0.88	0.79	0.92	0.54	0.84	0.81	0.58	0.62	0.605
2991	52	0.63	0.5	0.69	0.71	0.43	0.53	0.65	0.57	0.563
2992	82	0.71	0.91	0.97	0.96	0.88	0.92	0.73	0.79	0.816
2993	67	0.71	0.68	0.86	0.71	0.78	0.64	0.73	0.7	0.688

Table B.4 *Examinee Attribute Mastery Patterns (Step 3)*

ID	Raw score	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_{12}$	$\hat{\pi}_{23}$	$\hat{\pi}_{14}$	$\hat{\pi}_{124}$	$\hat{\pi}_{1234}$
1	77	1	1	1	1	1	1	1	1	1
2	79	1	1	1	1	1	1	1	1	1
3	79	1	1	1	1	1	1	1	1	1
4	53	1	0	0	0	0	0	0	0	0
5	68	1	1	1	0	0	0	0	0	0
6	77	1	1	1	1	1	1	1	1	1
7	67	0	1	1	0	0	1	0	0	1
8	37	0	0	0	0	0	0	0	0	0
9	75	1	1	1	1	1	1	1	1	1
10	41	0	0	0	0	0	0	0	0	0
11	68	1	1	1	0	1	1	0	0	0
12	36	0	0	0	0	0	0	0	0	0
13	63	1	0	1	1	0	0	1	0	0
14	61	1	1	0	0	1	0	0	0	0
15	69	0	0	1	1	1	0	0	0	0
16	42	0	0	0	0	0	0	0	0	0
17	66	1	1	0	0	1	0	0	1	0
18	62	1	1	0	0	1	0	0	1	0
19	77	1	1	1	1	1	1	1	1	1
20	63	0	1	1	0	0	1	0	0	0
:	**:**	**:**	**:**	**:**	**:**	**:**	**:**	**:**	**:**	**:**
2984	27	0	0	0	0	0	0	0	0	0
2985	73	1	1	1	1	1	1	1	1	1
2986	46	0	0	0	0	0	0	0	0	0
2987	50	0	0	0	0	0	0	0	0	0
2988	43	0	0	0	0	0	0	0	0	0
2989	70	1	1	1	1	1	1	0	0	0
2990	71	1	1	1	0	1	1	0	0	0
2991	52	0	0	0	1	0	0	0	0	0
2992	82	1	1	1	1	1	1	1	1	1
2993	67	1	1	1	1	1	0	1	1	0

APPENDIX C: SIMULATED (TRUE) ATTRIBUTE MASTERY PROBABILITIES, ATTRIBUTE MASTERY PATTERNS, AND RAW SCORES (FROM THE RESPONSE DATA) IN SIMULATION STUDY 1

	Attribute Mastery Probability				Attribute Mastery Pattern				Raw Score
id	A1	A2	A3	A4	A1	A2	A3	A4	(total:60)
1	0.0307	0.79772	0.6672	0.57368	0	1	0	0	22
2	0.725	0.97883	0.8832	0.79704	1	1	1	1	49
3	0.1926	0.12169	0.4687	0.23862	0	0	0	0	38
4	0.8355	0.80928	0.9519	0.78479	1	1	1	1	54
5	0.5994	0.73506	0.6598	0.77782	0	1	0	1	25
6	0.6174	0.49628	0.8571	0.66599	0	0	1	0	18
7	0.9133	0.84693	0.8007	0.80086	1	1	1	1	54
8	0.4447	0.92651	0.9205	0.19596	0	1	1	0	23
9	0.4657	0.44651	0.4082	0.43002	0	0	0	0	15
10	0.1316	0.18762	0.749	0.80472	0	0	1	1	27
11	0.8359	0.97984	0.7792	0.3121	1	1	1	0	45
12	0.8286	0.81866	0.7268	0.80168	1	1	1	1	58
13	0.5329	0.56335	0.301	0.25158	0	0	0	0	28
14	0.2048	0.93531	0.283	0.03471	0	1	0	0	22
15	0.6616	0.35837	0.8089	0.55035	0	0	1	0	19
16	0.9405	0.84085	0.7292	0.8469	1	1	1	1	52
17	0.3142	0.89857	0.8003	0.89466	0	1	1	1	28
18	0.7792	0.78615	0.6599	0.79013	1	1	0	1	45
19	0.0748	0.92525	0.8084	0.89115	0	1	1	1	21
20	0.7803	0.75835	0.853	0.8032	1	1	1	1	43
:	:	:	:	:	:	:	:	:	:
9991	0.3589	0.32564	0.435	0.13236	0	0	0	0	24
9992	0.9014	0.74277	0.8672	0.7997	1	1	1	1	48
9993	0.5908	0.30147	0.8858	0.13064	0	0	1	0	14
9994	0.6726	0.48113	0.2849	0.12537	0	0	0	0	19
9995	0.7036	0.02131	0.1052	0.62664	1	0	0	0	27
9996	0.5504	0.75655	0.9692	0.88003	0	1	1	1	36
9997	0.0896	0.5118	0.1627	0.10205	0	0	0	0	18
9998	0.7915	0.30302	0.5641	0.40766	1	0	0	0	16
9999	0.2931	0.97558	0.0114	0.24813	0	1	0	0	20
10000	0.8659	0.76826	0.3418	0.92896	1	1	0	1	45

APPENDIX D: AVERAGE CCR, RMSE, AND ASB OF THE 36 SIMULATED CONDITIONS FOR BIBP

Condition	Sample Size	Attribute Correlation	Attribute Difficulty	CCR	RMSE	ASB
1	100	$\mathbf{R_1}$ (no correlation)	Easy	0.727	0.128	0.034
2			Medium	0.803	0.135	0.016
3			Hard	0.877	0.141	-0.002
4		$\mathbf{R_2}$ (low)	Easy	0.729	0.129	0.034
5			Medium	0.815	0.135	0.014
6			Hard	0.885	0.142	-0.004
7		$\mathbf{R_3}$ (medium)	Easy	0.736	0.128	0.032
8			Medium	0.812	0.135	0.015
9			Hard	0.887	0.142	-0.005
10		$\mathbf{R_4}$ **(**high)	Easy	0.728	0.128	0.034
11			Medium	0.811	0.135	0.015
12			Hard	0.884	0.141	-0.001
13	300	$\mathbf{R_1}$ (no correlation)	Easy	0.729	0.128	0.034
14			Medium	0.805	0.135	0.016
15			Hard	0.888	0.141	-0.004
16		$\mathbf{R_2}$ (low)	Easy	0.732	0.128	0.033
17			Medium	0.812	0.135	0.014
18			Hard	0.884	0.142	-0.003
19		$\mathbf{R_3}$ (medium)	Easy	0.733	0.128	0.034
20			Medium	0.811	0.135	0.014
21			Hard	0.891	0.143	-0.005
22		$\mathbf{R_4}$ **(**high)	Easy	0.735	0.128	0.033
23			Medium	0.809	0.135	0.015
24			Hard	0.885	0.142	-0.003
25	500	$\mathbf{R_1}$ (no correlation)	Easy	0.746	0.128	0.034
26			Medium	0.821	0.136	0.012
27			Hard	0.894	0.142	-0.005
28		$\mathbf{R_2}$ (low)	Easy	0.746	0.130	0.030
29			Medium	0.823	0.136	0.011
30			Hard	0.902	0.143	-0.007
31		$\mathbf{R_3}$ (medium)	Easy	0.746	0.129	0.030
32			Medium	0.824	0.136	0.011
33			Hard	0.900	0.143	-0.008
34		$\mathbf{R_4}$ **(**high)	Easy	0.734	0.128	0.034
35			Medium	0.815	0.136	0.013
36			Hard	0.901	0.143	-0.008
Total				0.812	0.136	0.014

APPENDIX E: AVERAGE CCR, RMSE, AND ASB OF THE 36 SIMULATED CONDITIONS FOR DINA

Condition	Sample Size	Attribute Correlation	Attribute Difficulty	CCR	RMSE	ASB
1	100	$\mathbf{R_1}$ (no correlation)	Easy	0.941	0.237	-0.121
2			Medium	0.882	0.271	-0.027
3			Hard	0.646	0.363	-0.112
4		$\mathbf{R_2}$ (low)	Easy	0.863	0.303	-0.156
5			Medium	0.862	0.287	-0.020
6			Hard	0.582	0.380	-0.189
7		$\mathbf{R_3}$ (medium)	Easy	0.946	0.231	-0.138
8			Medium	0.894	0.279	-0.023
9			Hard	0.674	0.351	-0.078
10		$\mathbf{R_4}$ **(**high)	Easy	0.954	0.237	-0.131
11			Medium	0.914	0.259	-0.045
12			Hard	0.843	0.316	0.069
13	300	$\mathbf{R_1}$ (no correlation)	Easy	0.963	0.218	-0.128
14			Medium	0.884	0.262	-0.026
15			Hard	0.764	0.296	-0.012
16		$\mathbf{R_2}$ (low)	Easy	0.952	0.221	-0.124
17			Medium	0.872	0.271	-0.039
18			Hard	0.787	0.291	0.015
19		$\mathbf{R_3}$ (medium)	Easy	0.956	0.219	-0.125
20			Medium	0.889	0.264	-0.014
21			Hard	0.828	0.289	0.057
22		$\mathbf{R_4}$ **(**high)	Easy	0.964	0.223	-0.114
23			Medium	0.921	0.265	-0.002
24			Hard	0.862	0.296	0.090
25	500	$\mathbf{R_1}$ (no correlation)	Easy	0.961	0.219	-0.126
26			Medium	0.883	0.262	-0.025
27			Hard	0.748	0.295	-0.019
28		$\mathbf{R_2}$ (low)	Easy	0.933	0.229	-0.127
29			Medium	0.843	0.287	-0.045
30			Hard	0.770	0.290	0.005
31		$\mathbf{R_3}$ (medium)	Easy	0.957	0.222	-0.123
32			Medium	0.894	0.263	-0.011
33			Hard	0.808	0.290	0.043
34		$\mathbf{R_4}$ **(**high)	Easy	0.968	0.219	-0.116
35			Medium	0.919	0.268	0.007
36			Hard	0.872	0.294	0.091
Total				0.867	0.271	-0.051

REFERENCES:

Almond, R. G., Steinberg, L. S., & Mislevy, R. J. (2003). A framework for reusing assessement components. In H. Yanai, A. Okada, K. Shigemasu, Y. Kano, & J. J. Meulman. (Eds.), *New Development in Psychometrics.* Tokyo, Japan: Springer.

Berger, M. (1982). The scientific approach to intelligence: An overview of its history with special reference to mental speed. In H. J. Eysenck (Ed.), *A Model for Intelligence* (pp. 13-43). Berlin, New York: Springer-Verlag.

Bejar, I. I. (1985). Speculations on the future of test design. In S. E. Embretson (Ed.), *Test design: Contribution from education and psychometrics* (pp 279-294). New York Academic Press.

Bejar, I. I. (2008). Model based item generation: A review of recent research. In Embretson, S. E. (ed.) *New directions in measuring psychological constructs with model-base approaches.* Washington, D.C.: American Psychological Association Books.

Bejar, I. I., Lawless, R. R., Morley, M. E., Wagner, M. E., Bennett, R. E. and Revuelta, J. (2003). A feasibility study of on-the-fly item generation in adaptive testing. *Journal of Technology, Learning, and Assessment* 2, Available from http://www.jtla.org.

Bejar, I. I., & Yocom, P. (1986). *A generative approach to the development of hidden-figure items* (Research Report No. RR-86-20-ONR). Princeton, NJ: Educational Testing Service.

Bergstrom, B., Gershon, R., & Lunz, M. E. (1994). *Computerized Adaptive Testing Exploring Examinee Response Time Using Hierarchical Linear Modeling*. Paper presented at the annual meeting National Council on Measurement in Education New Orleans, LA: April.

Birenbaum, M., Kelly, A. E., & Tatsuoka, K. K. (1993). Diagnosing knowledge states in algebra using the rule-space model. *Journal for Research in Mathematics Education, 24*, 442–459.

Bloom, B. (1956). *Taxonomy of Educational Objectives, Handbook 1: Cognitive Domain.* New York: Longman.

Bolt, D. M. & Lall, V. F. (2003). Estimation of Compensatory and Noncompensatory Multidimensional Item Response Models Using Markov Chain Monte Carlo. *Applied Psychological Measurement, 27*, 395–414.

Brown, J. S., & Burton, R. R. (1978). Diagnostic models for procedural bugs in basic mathematics skills. *Cognitive Science, 2,* 155-192.

Burke, M. J., & Henson, R. (2008). *LCDM user's manual.* Greensboro: University of North Carolina at Greensboro.

Chipman, S. F., Nichols, P. D., & Brennan, R. L. (1995). Introduction. In P. D. Nichols, S. F. Chipman & R. L. Brennan (Eds.) *Cognitively diagnostic assessment* (pp. 1-18). Hillsdale, NJ: Erlbaum.

DiBello, L. V., Stout, W. F., & Roussos, L. A. (1993). *Unified cognitive/psychometric Diagnosis foundation.* Manuscript submitted for publication.

DiBello, L. V., Stout, W. F., & Roussos, L. A. (1995). Unified cognitive/psychometric diagnostic assessment likelihood-based classification techniques. In P. D. Nichols, S. F. Chipman & R. L. Brennan (Eds.), *Cognitively diagnostic assessment* (pp. 361-389). Hillsdale, NJ: Erlbaum.

DiBello, L. V., Roussos, L. A., & Stout, W. F. (2007). Review of cognitive diagnostic assessment and a summary of psychometric models. In C. R. Rao & S. Sinharay (Eds.), *Handbook of statistics: Volume 26, psychometrics* (pp. 979-1030). Amsterdam: Elsevier.

Embretson, S. E. (1983). Construct validity: Construct representation versus nomothetic span. *Psychological Bulletin, 93,* 179-197.

Embretson, S. E. (1984). A general multicomponent latent trait model for response processes. *Psychometrika* 49, 175-186

Embretson, S. E. (1985a). *Test design: Developments in psychology and psychometrics.* New York: Academic Press.

Embretson, S. E. (1985b). Studying intelligence with test theory models. *Current Topics in Human Intelligence, 1,* 98-140.

Embretson, S. E. (1990). Diagnostic testing by measuring learning processes: Psychometric considerations for dynamic testing. In N. Frederiksen, R. Glaser, A. Lesgold, & M. Shafto (Eds.), *Diagnostic monitoring of skill and knowledge acquisition.* (pp. 407-432). Hillsdale, NJ, England: Lawrence Erlbaum Associates, Inc.

Embretson, S. E. (1991). A multidimensional latent trait model for measuring learning and change. *Psychometrika, 56(3),* 495-516.

Embretson, S. E. (1993). Psychometric models for learning and cognitive processes. In N. Frederiksen, R. J. Mislevy & I. I. Bejar (Eds.), *Test theory for a new generation of tests* (pp. 125-150). Hillsdale, NJ: Erlbaum.

Embretson, S. E. (1994). Application of cognitive design systems to test development. In C. R. Reynolds (Ed.), *Cognitive assessment: A multidisciplinary perspective (*pp. 107-135). New York: Plenum Press.

Embretson, S. E. (1998). A cognitive design system approach to generating valid tests: Application to abstract reasoning. *Psychological Methods* 3, 300-396.

Embretson, S. E. (1999). Generating items during testing: Psychometric issues and models. *Psychometrika* 64, 407-433.

Embretson, S. E. (1995a). Developments toward a cognitive design system for psychological tests. In D. J. Lupinsky & R. V. Dawis (Eds.), *Assessing individual differences in human behavior* (pp. 17-48). Palo Alto, CA: Davies-Black Publishing.

Embretson, S. E. (1995b). The role of working memory capacity and general control processes in intelligence. *Intelligence, 20*, 169-189.

Embretson, S. E. (1995c). A measurement model for linking individual change to processes and knowledge: Application to mathematical reasoning. *Journal of Educational Measurement, 32,* 277-294.

Embretson, S. E. (1998a). A cognitive design system approach to generating valid tests: Application to abstract reasoning. *Psychological Methods, 3*, 300-396.

Embretson, S. E. (1998b, August). *Modifiability in lifespan development: Multidimensioal Rasch model for learning and change.* Paper presented at the meeting of the American Psychological Association, San Francisco, CA.

Embretson, S. E. (1999). Generating items during testing: Psychometric issues and models. *Psychometrika, 64*, 407-433.

Embretson, S. E. (2000). Multidimensional measurement from dynamic tests: Abstract reasoning under stress. *Multivariate Bahavioral research, 35 (4),* 505-542.

Embretson, S.E. (2002). Generating abstract reasoning items with cognitive theory. In Irvine, S, & Kyllonen, P. (Eds.) *Generating items for cognitive tests: Theory and Practice.* Mahwah, NJ:Erlbaum.

Embretson, S. E. (2003). Cognitive models for psychometric properties of GRE quantitative items. *ETS Progress Report for Research Project.*

Embretson, S. E. (2004a). The second century of ability testing:some predictions and speculations. *Measurement: Interdisciplinary research and perspectives, 2 (1),* 1-32.

Embretson, S. E. (2004b). *A cognitive model of mathematical reasoning*. Paper presented at the annual meeting of the Society for Multivariate Experimental Psychology. Naples, FL: October.

Embretson, S. E. (2005). Measuring human intelligence with artificial intelligence: Adaptive item generation. In R. J. Sternberg, & J. E. Pretz (Eds.), *Cognition and Intelligence: Identifying the mechanisms of the mind.* New York, NY: Cambridge University Press.

Embretson, S. E. (2006). Item Difficulty. Georgia Institute of Technology: Lectures in Psychology 7303, Psychometric Theory.

Embretson, S. E. (2007). Construct Validity: A Universal Validity System or Just Another Test Evaluation Procedure? *Educational Researcher, 36,* 449 - 455.

Embretson, S. E. & Daniel, R. C. (2008). Understanding and quantifying cognitive complexity level in mathematical problem solving Items. *Psychology Science.*

Embretson, S. E., Fultz, J., & Dayl, N. (1989). The influence of paragraph comprehension components on test validity. In R. F. Dillon, F. Ronna, J. Pellegrino, & W. James (Eds.), *Testing: Theoretical and applied perspectives.* (pp. 36-65). New York, NY, England: Praeger Publishers.

Embretson, S. E., & Gorin, J. (2001). Improving construct validity with cognitive psychology principles. *Journal of Educational Measurement, 38 (4),* 343-368.

Embretson, S. E. & McCollam, K. M. (2000). A multicomponent Rasch model for covert processes. In M. Wilson & G. Engelhard (Eds.), *Objective Measurement V.* Norwood, NJ: Ablex.

Embretson, S. E., & Prenovost, L. K. (2000). Dynamic cognitive testing: What kind of information is gained by measuring response time and modifiability? *Educational and Psychological Measurement, 60 (6),* 837-863.

Embretson, S.E., & Reise, S.P., (2000). *Item Response Theory for Psychologists.* Mahwah, New Jersey: Lawrence Erlbaum Associates, publishers.

Embretson, S. E., Schneider, L. M., & Roth, D. L. (1986). Multiple processing strategies and the construct validity of verbal reasoning tests. *Journal of Educational Measurement, 23 (1),* 13-32.

Embretson, S. E., & Yang, X. (2006a). Multicomponent latent trait models for complex tasks. *Journal of Applied Measurement, 7 (3),* 540-557.

Embretson, S. E. & Yang, X. (2007) Automatic item generation and cognitive psychology. In C. R. Rao & S. Sinharay (Eds.), *Handbook of statistics: Volume 26, psychometrics* (pp. 747-767). Amsterdam: Elsevier.

Embretson, S. E. & Yang, X. (2008). *Multicomponent latent trait model for cognitive diagnosis.* Paper presented at annual meeting of Psychometric Society, University of New Hampsire, NH.

Fischer, G. H. (1973). The linear logistic test model as an instrument in educational research. *Acta Psychologica* 37, 359-374.

Fischer, G. H. and Ponocny, I. (1994). An extension of the partial credit model with an application to the measurement of change. *Psychometrika* 59, 177-192.

Gierl, M. J., & Leighton, J. P. (2007). Directions for future research in cognitive diagnostic assessment. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.

Gierl, M .J., Leighton, J. P., & Tan, X. (2006). Evaluating DETECT classification accuracy and consistency when data display complex structure. *Journal of Educational Measurement. 43.* 265-289.

Gierl, M. J., Leighton, J. P., & Hunka, S. (2007). Using the Attribute Hierarchy Method to make diagnostic inferences about examinees' cognitive skills. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.

Gitomer, D. H., & Yamamoto, K. (1991). Performance modeling that integrates latent trait and class theory. *Journal of Educational Measurement, 28*, 173-189.

Gorin, J. S. (2007). Test construction and diagnostic testing. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.

Gorin, J. S. (2007). Reconsidering Issues in Validity Theory. *Educational Researcher, 36*, 456 - 462.

Gorin, J. S. & Embretson, S. E. (2006). Item difficulty modeling of paragraph comprehension items. *Applied Psychological Measurement, 30 (5),* 394-411.

Hartz, S. (2002). *A Bayesian framework for the Unified Model for assessing cognitive abilities: Blending theory with practice*. Unpublished doctoral dissertation, University of Illinois at Urbana-Champaign.

Hartz, S., Roussos, L., & Stout, W. (2002). Skills diagnosis: Theory and practice. User Manual for Arpeggio software. ETS.

Henson, R. & Douglas, J. (2005). Test construction for cognitive diagnosis. *Applied Psychological Measurement, 29*, 262-277.

Henson, R., Stout, W., Douglas, J., He, X., & Roussos, L. (2006). Cognitive Diagnostic Attribute Level Discrimination Indices. *Unpublished ETS Project Report*.

Jiang, H. (1996). Applications of Computational Statistics in Cognitive Diagnosis and IRT Modeling. Doctoral thesis, The University of Illinois at Urbana-Champaign.

Junker, B. W. & Sijtsma, K. (2001). Cognitive assessment models with few assumptions, and connects with non-parametric IRT. *Applied Psychological Measurement, 25*, 258-272.

Kim, J., & Bolt, D. M. (2007). An NCME instructional module on estimating item response theory models using Markov chain Monte Carlo methods. *Educational Measurement: Issues and Practice*, Winter 2007, 38-51.

Leighton, J. P., Gierl, M. J., & Hunka, S. (2004). The Attribute Hierarchy Method for Cognitive Assessment: A Variation on Tatsuoka's Rule-Space Approach. *Journal of Educational Measurement, 41*, 205-236.

Leighton, J. P.& Gierl, M. J.(2007). Verbal Reports as Data for Cognitive Diagnostic Assessment. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.

Leighton, J. P., & Gierl, M. J. (2007). Defining and evaluating models of cognition used in educational measurement to make inferences about examinees' thinking processes. *Educational Measurement: Issues and Practice, 26*(2), 3–16.

McGlohen, M. K. (2004). *The application of a cognitive diagnosis model via an analysis of a large-scale assessment and a computerized adaptive testing administration*, Unpublished doctoral dissertation, University of Texas at Austin.

McGlohen, M. K., Chang, H. H., & Wills, J. T. (2004). *Combining Computer Adaptive Testing Technology with Cognitively Diagnostic Assessment*, Paper presented at the annual meeting of the National Council on Measurement in Education, San Diego, CA.

McKinley, R. L., & Reckase, M. D. (1982). *The use of the general Rasch model with multidimensional item response data* (Research Rep. ONR 82-1). Iowa City IA: American College Testing.

Messick, S. (1989). Validity. In R. L. Linn (Ed.), *Educational measurement* (3rd ed., pp. 1-103). New York: American Council on Education/Macmillan.

Messick, S. (1995). Validity of psychological assessment: Validation of inferences from persons' responses and performances as scientific inquiry into score meaning. *The American Psychologist, 50,* 741 -749.

Mislevy, R. J. (1994). Evidence and inference in educational assessment. *Psychometrika, 59,* 439-483.

Mislevy, R. J. (1995). Probability-Based Inference in Cognitive Diagnosis. In P. D. Nichols, S. F. Chipman & R. L. Brennan (Eds.) *Cognitively diagnostic assessment* (pp. 43-72). Hillsdale, NJ: Erlbaum.

Mislevy, R. J. (2006). Cognitive psychology and educational assessment. In R. L. Brennan (Ed.), *Educational measurement* (4th ed., pp. 257–305). Westport, CT: American Council on Education/Praeger

Mislevy, R. J. (2007). Validity by Design, *Educational Researcher, 36,* 463 - 469.

Mislevy, R. J., Steinberg, L. S., & Almond, R. G. (2002). Design and analysis in task-based language assessment. *Language Testing. Special Issue: Interpretations, intended uses, and designs in task-based language, 19(4),* 477-496.

Mislevy, R. J., Steinberg, L. S., & Almond, R. G. (2003). On the structure of educational assessments. *Measurement: Interdisciplinary Research and Perspectives, 1,* 3-67.

Mislevy, R. J., Almond, R. G., & Lukas, J. F. (2004). A brief introduction to Evidence-Centered Design. CSE Technical Report 632, The National Center for research on Evaluation, Standard, and Student Testing (CRESST), Center for the Study of Evaluation (SCE), UCLA, Los Angeles, CA.

Muthen, L.K., & Muthen, B.O. (1998). *Mplus User's Guide*. Los Angeles: Muthen & Muthen.

Newstead, S. E., Bradon, P., Handley, S. J., Dennis, I., & Evans, J. S. B. T. (2006). Predicting the difficulty of complex logical reasoning problems. *Thinking & Reasoning, 12*(1), 62-90.

Newstead, S. E., Pollard, P., Evans, J. S. T., & Allen, J. L. (1992). The source of belief bias effects in syllogistic reasoning. *Cognition, 45*(3), 257-284.

Patz, R. J., & Junker, B. (1999). A straightforward approach to Markov chain Monte Carlo methods for item response models. *Journal of Educational and Behavioral Statistics, 24,* 146-178.

Rasch, G. (1960). *Probabilistic models for some intelligence and attainment tests.*

Chicago: University of Chicago Press

Rijmen, F., & De Boeck, P. (2001). Propositional reasoning: The differential contribution of "rules" to the difficulty of complex reasoning problems. *Memory & Cognition, 29*(1), 165-175.

Reckase, M. D. (1985). The difficulty of test items that measure more than one ability. *Applied Psychological Measurement. 9.* 401-412.

Reckase, M. D. (1997). The past and future of multidimensional item response theory. *Applied Psychological Measurement, 21,* 25-36.

Reckase, M. D., & McKinley, R. L. (1991). The discriminating power of items that measure more than one dimension. *Applied Psychological Measurement, 15.* 361-373.

Roskam, E. E. (1987). Toward a psychometric theory of intelligence. In E. E. Roskam and R. Suck (Eds.), *Progress in Mathematical Psychology* (pp. 151-174). Amsterdam: North-Holland.

Rupp, A.A., & Mislevy, R. J. (2007). Cognitive Foundations of Structured Item Response Models. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.

Rupp, A. A., & Templin, J. T. (2008). Unique Characteristics of Diagnostic Classification Models: A Comprehensive Review of the Current State-of-the-Art. *Measurement: Interdisciplinary Research & Perspective, 6*(4), 219-262.

Roussos, L. A., DiBello, L. V., Stout, W., Hartz, S. M., Henson, R. A., & Templin, J. L. (2007). The fusion model skills diagnosis system. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.

Roussos, L. A., DiBello, L. V., Henson, R. A., Jang, E. E., & Templin, J. L. (2008). Skills diagnosis for education and psychology with IRT-based parametric latent class models. In S. Embretson & J. Roberts (Eds.), *New directions in psychological measurement with model-based approaches.* Washington, DC: American Psychological Association.

Snow, R. E., & Lohman, D. F. (1989). Implication of cognitive psychology for educational measurement. In R. L. Linn (Ed.), *Educational measurement* (3rd ed., pp. 263-331). New York: American Council on Education/Macmillan.

Snow, R. E., & Lohman, D. F. (1993). Cognitive psychology, new test design, and new test theory: An introduction. In N. Frederiksen, R. J. Mislevy & I. I. Bejar (Eds.), *Test theory for a new generation of tests* (pp. 1-18). Hillsdale, NJ: Erlbaum.

Spiegelhalter, D. J., Abrams, K. R., & Myles, J. P. (2004). *Bayesian Approaches to Clinical Trials and Health-Care Evaluation.* England: John Wiley & Sons, Ltd.

Spiegelhalter, D. J., Thomas, A., Best, N., & Lunn, D. (2003). *WINBUGS Version 1.4 User's manual* [Computer software manual]. Cambridge, UK: MRC Biostatistics Unit.

Tatsuoka, K. K. (1983). Rule space: An approach for dealing with misconceptions based on item response theory. *Journal of Educational Measurement, 20,* 34-38.

Tatsuoka, K. K. (1984). Caution indices based on item response theory. *Psychometrika, 49,* 94-110.

Tatsuoka, K. K. (1985). A probabilistic model for diagnosing misconceptions in the pattern classification approach. *Journal of Educational Statistics, 12,* 55-73.

Tatsuoka, K. K. (1990). Toward an integration of item-response theory and cognitive error diagnoses. In N. Frederiksen, R. L. Glaser, A. M. Lesgold, & M. G. Shafto (Eds.), *Diagnostic monitoring of skill and knowledge acquisition* (pp. 453-486). Hillsdale, NJ: Lawrence Erlbaum Associates.

Tatsuoka, K. K. (1995). Architecture of knowledge structures and cognitive diagnosis: A statistical pattern recognition and classification approach. In P. D. Nichols, S. F. Chipman & R. L. Brennan (Eds.) *Cognitively diagnostic assessment* (pp. 327-359). Hillsdale, NJ: Erlbaum.

Tatsuoka, K. K. (1996). Use of generalized person-fit indexes, Zetas for statistical pattern classification. *Applied Measurement in Education, 9,* 65-75.

Tatsuoka, K. K., & Tatsuoka, M. M. (1987). Bug distribution and pattern classification. *Psychometrika, 52(2),* 193-206.

Tatsuoka, K. K., & Tatsuoka, M. M. (1992). A psychometrically sound cognitive diagnostic model: Effect of remediation as empirical validity (research report). Princeton, NJ: Educational Testing Service.

Tatsuoka, K. K., & Tatsuoka, M. M. (1997). Computerized cognitive diagnostic adaptive testing: Effect on remedial instruction as empirical validation. *Journal of Educational Measurement, 34,* 3-20.

Tatsuoka, K. K., Corter, J. E., & Tatsuoka, C. (2004). Patterns of diagnosed

mathematical content and process skills in TIMSS-R across a sample of 20 countries. *American Educational Research Journal*, *41*, pp. 901- 926.

Templin, J. L., & Henson, R. A. (2006). Measurement of psychological disorders using cognitive diagnosis models. *Psychological Methods, 11(3)*, 287-305.

Templin, J. L., Henson, R. A., & Douglas, J. (2006). *General theory and estimation of cognitive diagnosis models: Using Mplus to derive model estimates.* 2007 National Council on Measurement in Education training session, Chicago, Illinois.

Templin, J. L., Henson, R. A., Templin, S. E., & Roussos, L. (2008). Robustness of Hierarchical Modeling of Skill Association in Cognitive Diagnosis Models. *Applied Psychological Measurement, 32*, 559-574.

Thissen, D. (2010). IRTPRO Beta Features and Operation [Computer software manual], IL: Scientific Software International.

Tippins, N. T., Beaty, J., Drasgow, F., Gibson, W. M., Pearlman, K., Segall, D. O., & Shepherd, W. (2006). Unproctored internet testing in employment settings. *Personnel Psychology*, 59, 189-225.

Wilson, M., & De Boeck, P. (2004). Descriptive and explanatory item response models. In P. De Boeck, & M. Wilson (Eds.), *Explanatory Item Response Models: A generalized linear and nonlinear approach.* New York: Springer.

Whitely, S. E. (1980). Multicomponent latent trait models for ability tests. *Psychometrika*, 45, 479-494.

Yamamoto, K. (1989). *HYBRID model of IRT and latent class models* (ETS Research Report RR-89-41). Princeton, NJ: Educational Testing Service (ERIC Document Reproduction Service No. ED 310161).

Yamamoto, K. (1995). *Estimating the effects of test length and test time on parameter estimation using the HYBRID model* (ETS Research Report RR-95-2). Princeton, NJ: Educational Testing Service (ERIC Document Reproduction Service No. ED 395035).

Yamamoto, K., & Everson, H.T. (1995). *Modeling the mixture of IRT and pattern responses by a modified HYBRID model* (ETS Research Report RR-95-16). Princeton, NJ: Educational Testing Service (ERIC Document Reproduction Service No. ED 395036).

Yang, X., & Embretson, S. E. (2007). Construct validity and cognitive diagnostic assessment. In J. Leighton., & M. Gierl (Eds.), *Cognitive Diagnostic Assessment for Education.* New York, NY: Cambridge University Press.